THE PERMUTAHEDRAL VARIETY, MIXED EULERIAN NUMBERS, AND PRINCIPAL SPECIALIZATIONS OF SCHUBERT POLYNOMIALS

PHILIPPE NADEAU AND VASU TEWARI

Abstract. We compute the expansion of the cohomology class of the permutahedral variety in the basis of Schubert classes. The resulting structure constants a_w are expressed as a sum of *normalized* mixed Eulerian numbers indexed naturally by reduced words of w. The description implies that the a_w are positive for all permutations $w \in S_n$ of length $n-1$, thereby answering a question of Harada, Horiguchi, Masuda and Park. We use the same expression to establish the invariance of a_w under taking inverses and conjugation by the longest word, and subsequently establish an intriguing cyclic sum rule for the numbers.

We then move toward a deeper combinatorial understanding for the a_w by exploiting in addition the relation to Postnikov's divided symmetrization. Finally, we are able to give a combinatorial interpretation for a_w when w is vexillary, in terms of certain tableau descents. It is based in part on a relation between the numbers a_w and principal specializations of Schubert polynomials.

Along the way, we prove results and raise questions of independent interest about the combinatorics of permutations, Schubert polynomials and related objects.

1. Introduction and statement of results

The (type A) complete flag variety $\mathrm{Flag}(n)$ has been an active area of study for many decades. In spite of its purely geometric origins, it interacts substantially with representation theory and algebraic combinatorics. By way of the intricate combinatorics involved in the study of its Schubert subvarieties, the study of $\mathrm{Flag}(n)$ poses numerous intriguing questions, most notably that of providing a combinatorial rule to compute the intersection numbers of Schubert varieties. The bridge between the geometry and topology of Schubert varieties and the associated algebra and combinatorics is formed in great part by the Schubert polynomials, relying upon seminal work of Borel [11] and Lascoux-Schützenberger [39], followed by influential work of Billey-Jockusch-Stanley [10] and Fomin-Stanley [23].

Hessenberg varieties are a relatively recent family of subvarieties of $\mathrm{Flag}(n)$ introduced by De Mari, Procesi and Shayman [18] with inspiration drawn from numerical analysis. Their study has also revealed a rich interplay between geometry, representation theory and combinatorics [5, 30, 60]. The last decade has seen an ever-increasing interest in their study with impetus coming from the study of chromatic quasisymmetric functions and the potential ramifications for the Stanley-Stembridge conjecture [28, 54, 55]. The study of the cohomology rings of Hessenberg varieties has been linked to the study of hyperplane arrangements and representations of the symmetric group [4, 3, 16, 27]. We refer the reader to Abe and Horiguchi's excellent survey article [2] and references therein for more details on the rich vein of mathematics surrounding Hessenberg varieties.

Key words and phrases. divided symmetrization, Peterson variety, permutahedral variety, Schubert polynomials, cohomology class, flag variety, mixed Eulerian numbers.

Recall that $\mathrm{Flag}(n)$ is the collection of nested subspaces $V_\bullet = ((0) \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n)$ with $\dim(V_i) = i$ for all $i \in [n] := \{1, \ldots, n\}$. A *Hessenberg function* $h : [n] \to [n]$ is a function satisfying that condition that $i \leq h(i)$ for all $i \in [n]$ and $h(i) \leq h(j)$ for all $1 \leq i < j \leq n$. Given an $n \times n$ matrix X and a Hessenberg function $h : [n] \to [n]$, the *Hessenberg variety* (in type A) associated with X and h is defined to be

$$Hess(X, h) := \{V_\bullet \in \mathrm{Flag}(n) \mid X \cdot V_j \subset V_{h(j)} \text{ for all } j \in [n]\}.$$

Fix $h = (2, 3, \ldots, n, n)$. The *permutahedral variety* Perm_n is the Hessenberg variety corresponding to this choice of h and X being a diagonal matrix with distinct entries along the diagonal. This variety is a smooth toric variety whose fan is the so-called braid fan comprising the Weyl chambers of the type A root system. The permutahedral variety appears in many areas in mathematics and is well studied [37, 19, 52], and is a key player in the Huh-Katz resolution of the Rota-Welsh conjecture in the representable case [31]. The *Peterson variety* Pet_n is the Hessenberg variety defined with the same h, and with X chosen to be the nilpotent matrix that has ones on the upper diagonal and zeros elsewhere. This variety has also garnered plenty of attention recently; see [17, 20, 29, 32, 33, 35, 53].

Both Perm_n and Pet_n are irreducible subvarieties of $\mathrm{Flag}(n)$ of complex dimension $n-1$. In fact both are *regular* Hessenberg varieties: this means that the matrix X is regular, i.e. has only one Jordan block attached to any eigenvalue. Perm_n corresponds to the regular semisimple case, and Pet_n to the regular nilpotent case. It is known that for a given h, all regular Hessenberg varieties have the same class in the rational cohomology $H^*(\mathrm{Flag}(n))$, see [1].

We let τ_n be this cohomology class for $h = (2, 3, \ldots, n, n)$, so we have $\tau_n = [\mathrm{Perm}_n] = [\mathrm{Pet}_n]$. The class τ_n belongs to $H^{(n-1)(n-2)}(\mathrm{Flag}(n))$, and we consider its Schubert class expansion

$$\tau_n = \sum_{w \in \mathrm{S}'_n} a_w \sigma_{w_o w}. \tag{1.1}$$

Here S'_n denotes the set of permutations in S_n of length $n-1$.

The main goal of this article is to develop our understanding of the coefficients a_w in (1.1), which are known to be nonnegative integers from geometry. We unearth interesting connections between these numbers and the combinatorics of reduced words, principal specializations of Schubert polynomials, enumeration of flagged tableaux, as well as discrete-geometric notions, namely mixed volumes of hypersimplices. As a consequence, *we also obtain certain properties of the a_w that we do not know geometric reasons for.* It is worth emphasizing here that Anderson and Tymoczko [5] already give an expansion of τ_n that involves multiplying Schubert polynomials. As stated earlier, providing a combinatorial rule for this is a notoriously hard open problem in general. Hence, one is led to approach the question of providing a meaningful perspective on the a_w via alternative means. To this end we bring together work of Klyachko [37] and Postnikov [50].

We proceed to state our main results. The reader is referred to Section 2 for undefined terminology. Our first main result states that the a_w are strictly positive, that is, the expansion in (1.1) has *full support.* This answers a problem posed by Harada et al [27, Problem 6.6].

Theorem 1.1. *For $w \in \mathrm{S}'_n$, we have that $a_w > 0$. Furthermore, the following symmetries hold.*

- $a_w = a_{w_o w w_o}$ *where w_o denotes the longest word in S_n.*
- $a_w = a_{w^{-1}}$.

Our proof of the positivity of a_w relies on an explicit formula obtained as a sum of certain mixed Eulerian numbers A_c normalized by $(n-1)!$. These numbers were introduced by Postnikov [50, Section 16] as mixed volumes of Minkowski sums of hypersimplices. Curiously, while geometry says that the a_w are nonnegative integers, our formula expresses them as a sum of positive rational numbers. The fact that this sum is indeed integral hints at deeper reasons, which is what we explore subsequently.

Any permutation has a natural factorization into indecomposable permutations acting on disjoint intervals, where $u \in \mathrm{S}_p$ is called indecomposable if the image of $[i]$ does not equal $[i]$ for $i = 1, \ldots, p-1$; see Section 5.2 for precise definitions. One may rotate such blocks, thus giving rise to cyclic shifts of the permutation w. Given $w \in \mathrm{S}'_n$, let $w = w^{(1)}, w^{(2)}, \ldots, w^{(k)}$ be its cyclic shifts. Let us denote the set of reduced words of w by $\mathrm{Red}(w)$.

Our next chief result is a *cyclic sum rule*:

Theorem 1.2. *For $w \in \mathrm{S}'_n$ and with the notation just established we have that*

$$\sum_{1 \le i \le k} a_{w^{(i)}} = |\mathrm{Red}(w)|.$$

Theorem 1.2 hints at a potential refinement of the set of reduced words of w that would provide a combinatorial interpretation to the a_w. While we do not have such an interpretation in general, we obtain interpretations for important classes of permutations; we describe our results next.

Divided symmetrization is a linear form which acts on the space of polynomials in n indeterminates of degree $n-1$. This was introduced by Postnikov [50] in the context of computing volume polynomials of permutahedra. In its most general form, this operator sends a polynomial $f(x_1, \ldots, x_n)$ to a symmetric polynomial $\langle f(x_1, \ldots, x_n) \rangle_n$ as follows:

$$\langle f(x_1, \ldots, x_n) \rangle_n \coloneqq \sum_{w \in \mathrm{S}_n} w \cdot \left(\frac{f(x_1, \ldots, x_n)}{\prod_{1 \le i \le n-1} (x_i - x_{i+1})} \right), \tag{1.2}$$

where S_n acts by permuting variables. For homogeneous f of degree $n-1$, its divided symmetrization $\langle f \rangle_n$ is a scalar, and it is in this context that are results are primarily set. A computation starting with the Anderson-Tymoczko class of the Peterson variety [5] leads us to following conclusion already alluded to in the prequel [48] to this article — *for $w \in \mathrm{S}'_n$, we have that $a_w = \langle \mathfrak{S}_w \rangle_n$.* We are thus able to leverage our earlier work to obtain a better handle on the a_w.

We introduce a class of permutations in S'_n for which the corresponding a_w are particularly nice. We refer to these permutations as Łukasiewicz permutations in view of how they are defined. The set of Łukasiewicz permutations has cardinality given by the $(n-1)$-th Catalan number. A characteristic feature of these permutations is that a Schubert polynomial indexed by any such permutation is a sum of Catalan monomials (see [48]), and thus we have our next result.

Theorem 1.3. *For $w \in \mathcal{LP}_n$, we have that*

$$a_w = \mathfrak{S}_w(1, \ldots, 1).$$

In particular, a_w equals the number of reduced pipe dreams for any Łukasiewicz permutation $w \in \mathrm{S}'_n$.

In particular it follows that for 132-avoiding and 213-avoiding permutations $w \in \mathrm{S}'_n$, we have that $a_w = 1$. Another special case concerns Coxeter elements, for which $\mathfrak{S}_w(1, \ldots, 1)$ can be expressed as the number of permutations in S_{n-1} with a given descent set depending on w.

Our final results concerns the important class of permutations known as vexillary permutations, starting with the larger class of quasiindecomposable permutations. To state our results we need some more notation. Permutations of the form $1^a \times u \times 1^b$ for u indecomposable and $a, b \geq 0$, are said to be *quasiindecomposable*. Here $1^a \times u \times 1^b$ denotes the permutation obtained from u by inserting a fixed points at the beginning and b fixed points at the end.

Set $\nu_u(j) \coloneqq \mathfrak{S}_{1^j \times u}(1, 1, \dots)$ for $j \geq 0$.

Theorem 1.4. *Let $u \in \mathrm{S}_{p+1}$ be an indecomposable permutation of length $n-1$. We have that*

$$\sum_{j \geq 0} \nu_u(j) t^j = \frac{\displaystyle\sum_{m=0}^{n-p-1} a_{1^m \times u \times 1^{n-p-1-m}} t^m}{(1-t)^n},$$

We now come to our last result, which is of independent interest, making no mention of the numbers a_w. We establish that in the case where u is a vexillary permutation, the quantity $\nu_u(j)$ is essentially the order polynomial of a model of (P, ω)-*partitions* for appropriately chosen poset P and labeling ω. We refer the reader to Section 7 for precise details.

Theorem 1.5. *Let $u \in \mathrm{S}_{p+1}$ be an indecomposable vexillary permutation with shape $\lambda \vdash n-1$. Then there exist a labeling ω_u of λ and an integer $N_u \geq 0$ such that*

$$\sum_{j \geq 0} \nu_u(j) t^j = \frac{\displaystyle\sum_{T \in \mathrm{SYT}(\lambda)} t^{\mathrm{des}(T;\omega_u) - N_u}}{(1-t)^n},$$

where $\mathrm{SYT}(\lambda)$ denotes the set of standard Young tableaux of shape λ.

In the case u is indecomposable Grassmannian (respectively dominant), the statistic $\mathrm{des}(T; \omega_u)$ in the statement of Theorem 1.4 coincides with the usual descent (respectively ascent) statistic on standard Young tableaux for the appropriate choice of ω_u.

Outline of the article: Section 2 provides the necessary background on basic combinatorial notions attached to permutations, the cohomology of the flag variety, and some important properties of Schubert polynomials. Section 3 provides two perspectives on computing a_w, the first via Klyachko's investigation of the rational cohomology ring of Perm_n, and the second via Postnikov's divided symmetrization and a formula due to Anderson and Tymoczko. Section 4 introduces the mixed Eulerian numbers and surveys several of their properties, including a recursion that uniquely characterizes them. It also discusses Petrov's probabilistic take on these numbers. In Section 5, we use results of the preceding section to establish Theorems 1.1, 1.2 and 1.4. Section 6 discusses combinatorial interpretations for the a_w in special cases. In particular, we discuss the case of Łukasiewicz permutations, Coxeter elements as well as Grassmannian permutations, proving 1.3 in particular. Section 7 establishes our most general result as far as combinatorial interpretations go, by providing a complete understanding of the a_w for vexillary w through Theorem 1.5. We conclude with various remarks on further avenues and questions in Section 8.

2. Preliminaries

2.1. **Permutations.** We denote by S_n the group of permutations of $\{1,\dots,n\}$. We write an element w of S_n in one line notation, that is, as the word $w(1)w(2)\cdots w(n)$. The permutation $w_o = w_o^n$ is the element $n(n-1)\cdots 21$.

Descents: An index $1 \leq i < n$ is a *descent* of $w \in \mathrm{S}_n$ if $w(i) > w(i+1)$. The set of such indices is the *descent set* $\mathrm{Des}(w) \subseteq [n-1]$ of w. Given a subset $S \subseteq [n-1]$, define $\beta_n(S)$ to be the number of permutations $w \in \mathrm{S}_n$ such that $\mathrm{Des}(w) = S$. If $n = 4$ and $S = \{1,3\}$, one has $\beta_4(S) = |\{2143, 3142, 4132, 3241, 4231\}| = 5$.

Code and length: The *code* $\mathrm{code}(w)$ of a permutation $w \in \mathrm{S}_n$ is the sequence $(c_1, c_2, \dots, c_n)$ given by $c_i = |\{j > i \mid w(j) > w(i)\}|$. The map $w \mapsto \mathrm{code}(w)$ is a bijection from S_n to the set $C_n \coloneqq \{(c_1, c_2, \dots, c_n) \mid 0 \leq c_i \leq n-i,\ 1 \leq i \leq n\}$. The *shape* $\lambda(w)$ is the partition obtained by rearranging the nonzero elements of the code in nonincreasing order. The *length* $\ell(w)$ of a permutation $w \in \mathrm{S}_n$ is the number of *inversions*, i.e. pairs $i < j$ such that $w(i) > w(j)$. It is therefore equal to the sum $\sum_{i=1}^n c_i$ if $(c_1,\dots,c_n)$ is the code of w. The permutation $w = 3165274 \in \mathrm{S}_7$ has code $c(w) = (2,0,3,2,0,1,0)$, shape $\lambda(w) = (3,2,2,1)$ and length 8.

Let us recall the definition of the set S'_n, which naturally index the coefficients a_w:

$$\mathrm{S}'_n \coloneqq \{w \in \mathrm{S}_n \mid \ell(w) = n-1\}. \tag{2.1}$$

The cardinality of S'_n for $n = 1,\dots,10$ is $|\mathrm{S}'_n| = 1, 1, 2, 6, 20, 71, 259, 961, 3606, 13640$. The sequence occurs as number A000707 in the Online Encyclopaedia of Integer Sequences [56].

Pattern avoidance: Let $u \in \mathrm{S}_k$ and $w \in \mathrm{S}_n$ where $k \leq n$. An occurrence of the pattern u in w is a sequence $1 \leq i_1 < \cdots < i_k \leq n$ such that $u_r < u_s$ if and only if $w_{i_r} < w_{i_s}$. We say that w *avoids* the pattern u if it has no occurrence of this pattern and we refer to w as u-*avoiding*. For instance, 35124 has two occurrences of the pattern 213 at positions $1 < 3 < 5$ and $1 < 4 < 5$. It is 321-avoiding.

Reduced words: The symmetric group S_n is generated by the elementary transpositions $s_i = (i, i+1)$ for $i = 1,\dots,n-1$. Given $w \in \mathrm{S}_n$, the minimum length of a word $s_{i_1}\cdots s_{i_l}$ in the s_i's representing w is the length $\ell(w)$ defined above, and such a word is called a *reduced expression* for w. We denote by $\mathrm{Red}(w)$ the set of all *reduced words*, where $i_1\cdots i_l$ is a reduced word for w if $s_{i_1}\cdots s_{i_l}$ is a reduced expression of w. For the permutation $w = 3241$ of length 4, $\mathrm{Red}(w) = \{1231, 1213, 2123\}$. With these generators, S_n has a well-known Coxeter presentation given by the relations $s_i^2 = 1$ for all i, $s_is_j = s_js_i$ if $|j-i| > 1$ and $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$ for $i < n-1$. These last two sets of relations are called the *commutation relations* and *braid relations* respectively. Note that 321-avoiding permutations can be characterized as *fully commutative*: any two of their reduced expressions can be linked by a series of commutation relations [10].

The limit S_∞: One has natural monomorphisms $\iota_n : \mathrm{S}_n \to \mathrm{S}_{n+1}$ given by adding the fixed point $n+1$. One can then consider the direct limit of the groups S_n, denoted by S_∞: it is naturally realized as the set of permutations w of $\{1,2,3,\dots\}$ such that $\{i \mid w(i) \neq i\}$ is finite. Any group S_n thus injects naturally in S_∞ by restricting to permutations for which all $i > n$ are fixed points.

Most of the notions we defined for $w \in \mathrm{S}_n$ are well defined for S_∞. The code can be naturally extended to $w \in \mathrm{S}_\infty$ by defining $c_i = |\{j > i \mid w(j) > w(i)\}|$ for all $i \leq 1$. It is then a bijection between S_∞ and the set of infinite sequences $(c_i)_{i\geq 1}$ such that $\{i \mid c_i > 0\}$ is finite. The length

$\ell(w)$ is thus also well defined. Occurrences of a pattern $u \in \mathrm{S}_k$ are well defined in S_∞ if $u(k) \neq k$. Reduced words extend naturally.

2.2. **Flag variety, cohomology and Schubert polynomials.** Here we review standard material that can be found for instance in [24, 44, 15] and the references therein.

The flag variety $\mathrm{Flag}(n)$ is defined as the set of complete flags $V_\bullet = (V_0 = \{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n)$ where V_i is a linear subspace of $\mathbb{C}^n$ of dimension i for all i. For example, $V_\bullet^{std}, V_\bullet^{opp}$ are the standard and opposite flags given by $V_i^{std} = \mathrm{span}(e_1, \ldots, e_i)$ and $V_i^{opp} = \mathrm{span}(e_{n-i+1}, \ldots, e_n)$ respectively. $\mathrm{Flag}(n)$ has a natural structure of a smooth projective variety of dimension $\binom{n}{2}$. It admits a natural transitive action of GL_n via $g \cdot V_\bullet = (\{0\} \subset g(V_1) \subset g(V_2) \subset \cdots \subset \mathbb{C}^n)$. In fact $\mathrm{Flag}(n)$ is part of the family of *generalized flag varieties* G/B, with G a connected reductive group and B a Borel subgroup. In this context, $\mathrm{Flag}(n)$ corresponds to the type A case, with $G = GL_n$ and B the group of upper triangular matrices.

Given any fixed reference flag $V_\bullet^{ref}$, $\mathrm{Flag}(n)$ has a natural affine paving given by Schubert cells $\Omega_w(V_\bullet^{ref})$ indexed by permutations $w \in \mathrm{S}_n$. As algebraic varieties one has $\Omega_w(V_\bullet^{ref}) \simeq \mathbb{C}^{\ell(w)}$ where $\ell(w)$ is the length of w. By taking closures of these cells, one gets the family of *Schubert varieties* $X_w(V_\bullet^{ref})$.

The cohomology ring $H^*(\mathrm{Flag}(n))$ with rational coefficients is a well-studied graded commutative ring that we now go on to describe. It is known that to any irreducible subvariety $Y \subset \mathrm{Flag}(n)$ of dimension d can be associated a *fundamental class* $[Y] \in H^{n(n-1)-2d}(\mathrm{Flag}(n))$. In particular there are classes $[X_w(V_\bullet^{ref})] \in H^{n(n-1)-2\ell(w)}$. These classes do not in fact depend on $V_\bullet^{ref}$, and we write $\sigma_w \coloneqq [X_{w_o w}(V_\bullet^{ref})] \in H^{2\ell(w)}(\mathrm{Flag}(n))$. The affine paving by Schubert cells implies that these *Schubert classes* σ_w form a linear basis of $H^*(\mathrm{Flag}(n))$,

$$H^*(\mathrm{Flag}(n)) = \bigoplus_{w \in \mathrm{S}_n} \mathbb{Q}\sigma_w. \tag{2.2}$$

Now given Y irreducible of dimension d, we have an expansion of its fundamental class

$$[Y] = \sum_w b_w \sigma_w, \tag{2.3}$$

where the sum is over permutations of length $\ell(w_o) - d$. Then an important fact is that b_w *is a nonnegative integer*. Indeed, b_w can be interpreted as the number of points in the intersection of Y with $X_{w_o w}(V_\bullet^{ref})$ where $V_\bullet^{ref}$ is a *generic* flag.

One of the most important problems is to give a combinatorial interpretation to the coefficients when $Y = X_u(V_\bullet^{std}) \cap X_{w_o v}(V_\bullet^{opp})$ with $u, v \in \mathrm{S}_n$, that is Y is a *Richardson variety*. Indeed the coefficients b_w in this case are exactly the structure coefficients c_{uv}^w encoding the cup product in cohomology:

$$\sigma_u \cup \sigma_v = \sum_{w \in \mathrm{S}_n} c_{uv}^w \sigma_w. \tag{2.4}$$

2.3. **Borel presentation and Schubert polynomials.** Let $\mathbb{Q}[\mathbf{x}_n] \coloneqq \mathbb{Q}[x_1, \ldots, x_n]$ be the polynomial ring in n variables. We denote the space of homogeneous polynomials of degree $k \geq 0$ in $\mathbb{Q}[\mathbf{x}_n]$ by $\mathbb{Q}^{(k)}[\mathbf{x}_n]$. Let $\Lambda_n \subseteq \mathbb{Q}[\mathbf{x}_n]$ be the subring of symmetric polynomials in $x_1, \ldots, x_n$, and I_n be the ideal of $\mathbb{Q}[\mathbf{x}_n]$ generated by the elements $f \in \Lambda_n$ such that $f(0) = 0$. Equivalently, I_n

is generated as an ideal by the elementary symmetric polynomials $e_1, \dots, e_n$. The quotient ring $R_n = \mathbb{Q}[\mathbf{x}_n]/I_n$ is the *coinvariant ring.*

Let $\mathfrak{d}_i$ be the divided difference operator on $\mathbb{Q}[\mathbf{x}_n]$, given by

$$\mathfrak{d}_i(f) = \frac{f - s_i \cdot f}{x_i - x_{i+1}}. \tag{2.5}$$

Define the *Schubert polynomials* for $w \in \mathrm{S}_n$ as follows: $\mathfrak{S}_{w_o} = x_1^{n-1}x_2^{n-2}\cdots x_{n-1}$, while if i is a descent of w, let $\mathfrak{S}_{ws_i} = \mathfrak{d}_i\mathfrak{S}_w$. These are well defined since the $\mathfrak{d}_i$ satisfy the braid relations. For $w \in \mathrm{S}_n$, the Schubert polynomial $\mathfrak{S}_w$ is a homogeneous polynomial of degree $\ell(w)$ in $\mathbb{Q}[\mathbf{x}_n]$. In fact Schubert polynomials are well defined for $w \in \mathrm{S}_\infty$. Moreover, when $w \in \mathrm{S}_\infty$ runs through all permutations whose largest descent is at most n, the Schubert polynomials $\mathfrak{S}_w$ form a basis $\mathbb{Q}[\mathbf{x}_n]$.

Now consider the ring homomorphism

$$j_n : \mathbb{Q}[x_1, \dots, x_n] \to H^*(\mathrm{Flag}(n)) \tag{2.6}$$

given by $j_n(x_i) = \sigma_{s_i} - \sigma_{s_i-1}$ for $i > 1$ and $j_n(x_1) = \sigma_{s_1}$ (this is equivalent to the usual definition in terms of Chern classes). Then we have the following theorem, grouping famous results of Borel [11] and Lascoux and Schützenberger [39], see also [44, Section 3.6].

Theorem 2.1. *The map j_n is surjective and its kernel is I_n. Therefore $H^*(\mathrm{Flag}(n))$ is isomorphic as an algebra to R_n. Furthermore, $j_n(\mathfrak{S}_w) = \sigma_w$ if $w \in \mathrm{S}_n$, and $j_n(\mathfrak{S}_w) = 0$ if $w \in \mathrm{S}_\infty - \mathrm{S}_n$ has largest descent at most n.*

It follows immediately that the product of Schubert polynomials is given by the structure coefficients in (2.4): If $u, v \in \mathrm{S}_n$, then

$$\mathfrak{S}_u\mathfrak{S}_v = \sum_{w \in \mathrm{S}_n} c_{uv}^w \mathfrak{S}_w \mod I_n. \tag{2.7}$$

It is also possible to work directly in $\mathbb{Q}[\mathbf{x}_n]$ and not the quotient R_n: the coefficients c_{uv}^w are well defined for $u, v, w \in \mathrm{S}_\infty$, and one has

$$\mathfrak{S}_u\mathfrak{S}_v = \sum_{w \in \mathrm{S}_\infty} c_{uv}^w \mathfrak{S}_w. \tag{2.8}$$

2.4. **Expansion in Schubert classes and degree polynomials.** Given $\beta \in H^*(\mathrm{Flag}(n))$, let $\int \beta$ be the coefficient of σ_{w_o} in the Schubert class expansion. Then we have the natural *Poincaré duality* pairing on $H^*(\mathrm{Flag}(n))$ given by $(\alpha, \beta) \mapsto \int(\alpha \cup \beta)$. The Schubert classes are known to satisfy $\int \sigma_u \cup \sigma_v = 1$ if $u = w_o v$ and 0 otherwise, so that the pairing is nondegenerate. If $A, B \in \mathbb{Q}[\mathbf{x}_n]$ are such that $j_n(A) = \alpha, j_n(B) = \beta$, then one can compute the pairing explicitly by:

$$\int (\alpha \cup \beta) = \mathfrak{d}_{w_o}(AB)(0), \tag{2.9}$$

where the right hand side denotes the constant term in $\mathfrak{d}_{w_o}(AB)$.

The rest of this section is certainly well known to specialists, though perhaps not presented in this form. We simply point out that given a cohomology class, computing its expansion in terms of Schubert classes and its degree polynomial correspond to evaluating a given linear form on two different families of polynomials.

Let us fix $\alpha \in H^{n(n-1)-2p}(\mathrm{Flag}(n))$. Our main interest is to consider $\alpha = [Y]$ where Y is an irreducible closed subvariety of $\mathrm{Flag}(n)$ of dimension p. Associated to α is the linear form $\psi_\alpha : \beta \mapsto \int(\alpha \cup \beta)$ defined on $H^*(\mathrm{Flag}(n))$. It vanishes if β is homogeneous of degree $\neq 2p$, which leads to the following definition.

Definition 2.2. Given $\alpha \in H^{n(n-1)-2p}(\mathrm{Flag}(n))$ define the linear form $\phi_\alpha : \mathbb{Q}^{(p)}[\mathbf{x}_n] \to \mathbb{Q}$ by $\phi_\alpha(P) = \psi_\alpha(j_n(P))$ where j_n is the Borel morphism defined earlier.

Note that by definition, ϕ_α vanishes on $\mathbb{Q}^{(p)}[\mathbf{x}_n] \cap I_n$. For any polynomial $A, P \in \mathbb{Q}[\mathbf{x}_n]$ such that $j_n(A) = \alpha$, we have by (2.9) the expression

$$\phi_\alpha(P) = \mathfrak{d}_{w_o}(AP)(0). \tag{2.10}$$

The coefficient b_w in the expansion $\alpha = \sum_w b_w \sigma_w$ is given by

$$b_w = \phi_\alpha(\mathfrak{S}_{w_o w}) = \mathfrak{d}_{w_o}(\mathfrak{S}_{w_o w} A)(0). \tag{2.11}$$

Indeed $j_n(\mathfrak{S}_{w_o w}) = \sigma_{w_o w}$ by Theorem 2.1, and we use the duality of Schubert classes $\int \sigma_u \cup \sigma_v = 0$ unless $v = w_o u$ where it is 1.

The *degree polynomial* of α is defined by

$$\phi_\alpha((\lambda_1 x_1 + \cdots + \lambda_n x_n)^p),$$

see [27, 51]. It is a polynomial in $\lambda = (\lambda_1, \ldots, \lambda_n)$, where coefficients are given by applying ϕ_α to a monomial. When $\alpha = [Y]$ for a subvariety Y, and $\lambda \in \mathbb{Q}^n$ is a strictly dominant weight $\lambda_1 > \cdots > \lambda_n \geq 0$, $\phi_\alpha((\lambda_1 x_1 + \cdots + \lambda_n x_n)^p)$ gives the degree of Y in its embedding in $\mathbb{P}(V_\lambda)$ where V_λ denotes the irreducible representation of GL_n with highest weight λ.

The degree polynomials $\mathcal{D}_w(\lambda_1, \ldots, \lambda_n)$ of Schubert classes σ_w are studied in [51]. Note that if $\alpha = \sum_w b_w \sigma_w$ as above, then by linearity the degree polynomial of α is $\sum_w b_w \mathcal{D}_w(\lambda_1, \ldots, \lambda_n)$.

2.5. **Pipe dreams.** The *BJS formula* of Billey, Jockusch and Stanley [10] is an explicit nonnegative expansion of $\mathfrak{S}_w$ in the monomial basis:

$$\mathfrak{S}_w(x_1, \ldots, x_n) = \sum_{\mathrm{i} \in \mathrm{Red}(w)} \sum_{b \in C(\mathrm{i})} \mathbf{x}^b, \tag{2.12}$$

where $C(\mathrm{i})$ is the set of compositions $b_1 \leq \ldots \leq b_l$ such that $1 \leq b_j \leq i_j$, and $b_j < b_{j+1}$ whenever $i_j < i_{j+1}$. Additionally, $\mathbf{x}^b$ is the monomial $x_1^{b_1} \cdots x_l^{b_l}$.

The expansion in (2.12) has a nice combinatorial version with *pipe dreams* (also known as rc-graphs), which we now describe. Let $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ be the semi-infinite grid, starting from the northwest corner. Let (i, j) indicate the position at the ith row from the top and the jth column from the left. A *pipe dream* is a tiling of this grid with $+$'s (pluses) and 's (elbows) with a finite number of $+$'s. The *size* $|\gamma|$ of a pipe dream γ is the number of $+$'s.

Any pipe dream can be viewed as composed of *strands*, which cross at the $+$'s. Strands naturally connect bijectively rows on the left edge of the grid and columns along the top; let $w_\gamma(i) = j$ if the ith row is connected to the jth column, which defines a permutation $w_\gamma \in \mathrm{S}_\infty$.

Say that γ is *reduced* if $|\gamma| = \ell(w_\gamma)$; equivalently, any two strands of γ cross at most once. We let $\mathrm{PD}(w)$ be the number of reduced pipe dreams γ such that $w_\gamma = w$. Notice that if $w \in \mathrm{S}_n$ then the $+$'s in any $\gamma \in \mathrm{PD}(w)$ can only occur in positions (i,j) with $i+j<n$, so we can restrict the grid to such positions.

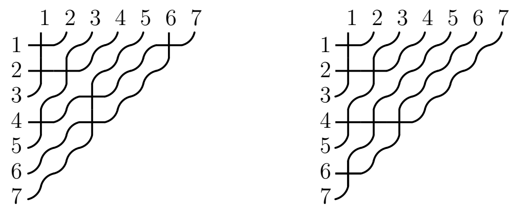


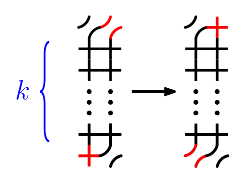
FIGURE 1. Two reduced pipe dreams with permutation $w_\gamma = 2417365$. On the right is the bottom pipe dream attached to this permutation.

Given $\gamma \in \mathrm{PD}(w)$, define $c(\gamma) \coloneqq (c_1, c_2, \ldots)$ where c_i is the number of $+$'s on the ith row of γ. Then the BJS expansion (2.12) can be rewritten as follows [10, 44]:

$$\mathfrak{S}_w = \sum_{\gamma \in \mathrm{PD}(w)} \mathbf{x}^{c(\gamma)}. \tag{2.13}$$

Given $w \in \mathrm{S}_\infty$, let $(c_1, c_2, \ldots) = \mathrm{code}(w)$. The *bottom pipe dream* $\gamma_w \in \mathrm{PD}(w)$ consists of $+'s$ in columns $1, \ldots, c_i$ for each row $i = 1, \ldots, n$; note that $c(\gamma_w) = \mathrm{code}(w)$.

A *ladder move* is a local operation on pipe dreams illustrated on the right: here k can be any nonnegative integer. When $k=0$ this is called a *simple* ladder move. The following result shows how to easily generate all pipe dreams attached to a given permutation.



Theorem 2.3. *([6, Theorem 3.7]) Let $w \in \mathrm{S}_n$. If $\gamma \in \mathrm{PD}(w)$, then γ can be obtained by a sequence of ladder moves from γ_w.*

Definition 2.4. For any $w \in \mathrm{S}_\infty$, define the *principal specialization* ν_w of the Schubert polynomials $\mathfrak{S}_w$ by $\nu_w = \mathfrak{S}_w(1, 1, \ldots)$.

By the expansion 2.13, one has the combinatorial interpretation

$$\nu_w = |\mathrm{PD}(w)|. \tag{2.14}$$

An alternative expression for ν_w is given by *Macdonald's reduced word identity* [43]

$$\nu_w = \frac{1}{\ell!} \sum_{\mathbf{i} \in \mathrm{Red}(w)} i_1 i_2 \cdots i_\ell. \tag{2.15}$$

A deeper study of Macdonald's reduced word identity and its generalizations has seen renewed interest recently and has brought forth various interesting aspects of the interplay between Schubert polynomials, combinatorics of reduced words, and differential operators on polynomials. We refer the reader to [9, 26, 62, 47] for more details. As we shall see in the next section, an expression rather reminiscent of the right hand side of (2.15) plays a key role in our quest to obtain the Schubert expansion for $\tau_n = [\mathrm{Perm}_n]$, and its appearance in this context begs for deeper explanation.

3. Formulas for a_w

Recall that we want to investigate the numbers a_w occurring in the Schubert class expansion

$$\tau_n = \sum_{w \in \mathrm{S}'_n} a_w \sigma_{w_o w} \in H^*(\mathrm{Flag}(n)).$$

Now τ_n is the class of the variety Perm_n, so by the classical results recalled in Section 2.2, we know that the a_w are nonnegative integers: namely a_w is the number of points in the intersection of Perm_n with a Schubert variety $X_{w_o w}(V_\bullet)$ where $V_\bullet$ is a *generic* flag.

In this section we use two approaches — the first due to Klyachko [37], the second due to Anderson-Tymoczko [5]— to arrive at algebraic expressions for the numbers a_w. These are given in Theorems 3.1 and 3.2 respectively, and both expressions will be exploited to extract various properties of the numbers a_w.

3.1. **Klyachko's approach.** We will extract our first expression from the results of [37]. Given $w \in \mathrm{S}_\infty$ of length ℓ, consider the polynomial in $\mathbb{Q}[x_1, x_2, \ldots]$:

$$\mathrm{M}_w(x_1, x_2, \ldots) := \sum_{\mathbf{i} = i_1 i_2 \cdots i_\ell \in \mathrm{Red}(w)} x_{i_1} x_{i_2} \cdots x_{i_\ell} = \sum_{\mathbf{i} \in \mathrm{Red}(w)} \mathbf{x}^{c(\mathbf{i})}, \tag{3.1}$$

where $c(\mathbf{i}) = (c_1, c_2, \ldots)$ and c_j is the number of occurrences of j in $\mathbf{i}$. If $w \in \mathrm{S}_n$, then M_w is a polynomial in $x_1, \ldots, x_{n-1}$. Notice that Macdonald's formula (2.15) states that

$$M_w(1, 2, \ldots) = \ell! \cdot \nu_w.$$

For $n \geq 3$, let $\mathbf{D}_n$ be the commutative $\mathbb{Q}$-algebra with generators $u_1, \ldots, u_{n-1}$ and defining relations

$$\begin{cases} 2u_i^2 = u_i u_{i-1} + u_i u_{i+1} \text{ for } 1 < i < n-1; \\ 2u_1^2 = u_1 u_2; \\ 2u_{n-1}^2 = u_{n-1} u_{n-2}. \end{cases}$$

Given $I = \{i_1 < \cdots < i_j\} \subset [n-1]$, define $u_I := u_{i_1} \cdots u_{i_j}$. Then the elements u_I, $I \subset [n-1]$ form a basis of $\mathbf{D}_n$. Given $U = \sum_I c_I u_I \in \mathbf{D}_n$, let $\int_{\mathbf{D}_n} U$ be the top coefficient $c_{[n-1]}$.

Theorem 3.1. *For any $w \in \mathrm{S}'_n$,*

$$a_w = \int_{\mathbf{D}_n} M_w(u_1, u_2, \ldots, u_{n-1}).$$

Proof. This is a light reformulation of Klyachko's work [37], specialized to type A. The rational cohomology ring of Perm_n is computed in this work. S_n acts on this ring, and the corresponding subring of invariants is shown to be the algebra $\mathbf{D}_n$ above. In this presentation, the fundamental class of Perm_n is the top element $u_{[n-1]}/(n-1)!$.

Now the embedding $\mathrm{Perm}_n \to \mathrm{Flag}(n)$ gives a pullback morphism $H^*(\mathrm{Flag}(n)) \to \mathbf{D}_n$, under which the image of the Schubert class σ_w is $M_w(u_1, u_2, \ldots, u_{n-1})/\ell(w)!$. Let $w \in \mathrm{S}'_n$. We have $a_w = \int \sigma_w \cup \tau_n = \int \sigma_w \cup [\mathrm{Perm}_n]$. By pulling back the computation to $\mathbf{D}_n$, we get the result. $\square$

We note that Klyachko was particularly interested in the case where w is Grassmannian, for which he gives a formula [37, Theorem 6] that is not manifestly positive. We will give a combinatorial interpretation for a_w, see Theorem 6.16, based on the approach from the next section.

3.2. **Anderson–Tymoczko's approach.** We have already encountered the operator of divided symmetrization $\langle\cdot\rangle_n$ in the introduction.

Theorem 3.2. *For any $w \in \mathrm{S}'_n$,*

$$a_w = \left\langle \mathfrak{S}_w(x_1, \ldots, x_n) \right\rangle_n. \tag{3.2}$$

We recall some of the relevant results from [5]. We consider a Hessenberg variety $\mathcal{H}(X,h)$ for h a Hessenberg function $[n] \to [n]$ and X a *regular* matrix. This means that X has exactly one Jordan block attached to each eigenvalue. Since regular Hessenberg varieties form a flat family [1], the class $\Sigma_h = [\mathcal{H}(X,h)] \in H^*(\mathrm{Flag}(n))$ does not depend on X.

By relating $\mathcal{H}(X,h)$ to a degeneracy locus for X regular semisimple, Anderson and Tymoczko [5] express Σ_h as a certain specialization of a *double Schubert polynomial* [44]. We identify $H^*(\mathrm{Flag}(n))$ and $R_n = \mathbb{Q}[\mathbf{x}_n]/I_n$ thanks to Theorem 2.1. The main result of [5] is

$$\Sigma_h = \mathfrak{S}_{w_h}(x_1, \cdots, x_n; x_n, \cdots, x_1) \mod I_n \tag{3.3}$$

$$= \prod_{\substack{1 \le i < j \le n \\ j > h(i)}} (x_i - x_j) \mod I_n. \tag{3.4}$$

where w_h is the permutation given by $\mathrm{code}(w_h^{-1}) = (n - h(1), \ldots, n - h(n))$. The simple product form above for double Schubert polynomial comes form the fact that w_h is a dominant permutation, cf. [44, Proposition 2.6.7].

Now in the case of $h = (2, 3, \ldots, n, n)$, we have that $\Sigma_h = \tau_n$ by definition and thus

$$\tau_n = \prod_{\substack{1 \le i < j \le n \\ j > i+1}} (x_i - x_j) \mod I_n.$$

Following the terminology of Section 2.4, consider the linear form ϕ_{τ_n} defined on $\mathbb{Q}^{(n-1)}[\mathbf{x}_n]$ by

$$\phi_{\tau_n}(P) = \mathfrak{d}_{w_o}(P \prod_{\substack{1 \le i < j \le n \\ j > h(i)}} (x_i - x_j))$$

We know that $\phi_{\tau_n}(\mathfrak{S}_w) = a_w$ by (2.11), so that Theorem 3.2 follows immediately from the next proposition.

Proposition 3.3. *For any $P \in \mathbb{Q}^{(n-1)}[\mathbf{x}_n]$,*

$$\phi_{\tau_n}(P) = \left\langle P \right\rangle_n.$$

Proof. Let Anti_n and Sym_n denote the antisymmetrizing operator $\sum_{\sigma \in \mathrm{S}_n} \epsilon(\sigma)\sigma$ and symmetrizing operator $\sum_{\sigma \in \mathrm{S}_n} \sigma$ acting on $\mathbb{Q}[\mathbf{x}_n]$ respectively. Here the action of the symmetric group permutes indeterminates, and $\epsilon(\sigma)$ denotes the *sign* of σ. Let Δ_n denote the usual Vandermonde determinant given by $\prod_{1 \le i < j \le n}(x_i - x_j)$.

One has $\mathfrak{d}_{w_o} = \frac{1}{\Delta_n} \mathrm{Anti}_n$ [44, Proposition 2.3.2] so that

$$\phi_{\tau_n}(P) = \frac{1}{\Delta_n}\,\mathrm{Anti}_n\left(P\prod_{1\le i<j\le n, j\neq i+1}(x_i-x_j)\right) = \frac{1}{\Delta_n}\,\mathrm{Anti}_n\left(\frac{P\Delta_n}{\prod_{1\le i\le n-1}(x_i-x_{i+1})}\right)$$
$$= \frac{\Delta_n}{\Delta_n}\mathrm{Sym}_n\left(\frac{P}{\prod_{1\le i\le n-1}(x_i-x_{i+1})}\right) = \left\langle P\right\rangle_n.$$

Here we used the fact that $\sigma(\Delta_n)=\epsilon(\sigma)\Delta_n$ between the first and second lines. □

Remark 3.4. There is an alternative way to prove Proposition 3.3 (equivalently, Theorem 3.2), which illuminates why the operator of divided symmetrization occurs in our context.

It is well-known that Perm_n is a smooth toric variety. Therefore its degree in the embedding $\mathbb{P}(V_\lambda)$ for λ strictly dominant is given by the (normalized) volume of its associated polytope. This polytope is the permutahedron with vertices given by permutations of $(\lambda_1,\dots,\lambda_n)$; see next section for more details. The volume was computed by Postnikov [50, Theorem 3.2] as a polynomial in $(\lambda_1,\dots,\lambda_n)$; his result is that the degree polynomial of $\tau_n=[\mathrm{Perm}_n]$ is $\left\langle(\lambda_1x_1+\cdots+\lambda_nx_n)^{n-1}\right\rangle_n$. Since this degree polynomial completely characterizes ϕ_{τ_n}, this proves Proposition 3.3.

4. Mixed Eulerian numbers

We turn our attention to an intriguing family of positive integers introduced by Postnikov [50]. These are the *mixed Eulerian numbers* $A_{c_1,\dots,c_n}$ indexed by weak compositions $c\coloneqq(c_1,\dots,c_n)$ where $\sum_{1\le i\le n}c_i=n-1$. We denote by $\mathcal{W}_n'$ the set of such compositions. Recall that a *weak composition* $(c_1,\dots,c_n)$ is simply a sequence of nonnegative integers. A *strong composition* $a=(a_1,\dots,a_p)$ is composed of positive integers, and we write $a\vDash N$ if $\sum_{1\le i\le p}a_i=N$. If $c=(0^{k-1},n-1,0^{n-k})$ for some $1\le k\le n$, then A_c equals the classical Eulerian number enumerating permutations in S_{n-1} with $k-1$ descents, which explains the name for the A_c in general.

We collect here various aspects of the mixed Eulerian numbers that shall play a key role in what follows.

We begin by explaining how they arise in Postnikov's work. Given $\lambda\coloneqq(\lambda_1\ge\cdots\ge\lambda_n)\in\mathbb{R}^n$, let P_λ be the *permutahedron* in $\mathbb{R}^n$ obtained by considering the convex hull of all points in the S_n-orbit of λ. Let $\mathrm{Vol}(P_\lambda)$ denote the usual $(n-1)$-dimensional volume of the polytope obtained by projecting P_λ onto the hyperplane defined by the n-th coordinate equaling 0.

By [50, Theorem 3.1], we have that

$$(n-1)!\mathrm{Vol}(P_\lambda)=\left\langle(\lambda_1x_1+\cdots\lambda_nx_n)^{n-1}\right\rangle_n \tag{4.1}$$

Setting $u_i=\lambda_i-\lambda_{i+1}$ for $1\le i\le n-1$, and $u_n=\lambda_n$, we have that

$$\sum_{1\le i\le n}\lambda_ix_i=\sum_{1\le i\le n}u_i(x_1+\cdots+x_i). \tag{4.2}$$

For brevity, set y_i equal to $x_1+\cdots+x_i$, and for $c=(c_1,\dots,c_n)$ define

$$y^c\coloneqq\prod_{1\le i\le n}y_i^{c_i}. \tag{4.3}$$

This given, rewrite (4.1) to obtain

$$\mathrm{Vol}(P_\lambda) = \sum_{c\in\mathcal{W}'_n} \left\langle y^c \right\rangle_n \frac{u_1^{c_1}\cdots u_n^{c_n}}{c_1!\cdots c_n!}. \tag{4.4}$$

We define the *mixed Eulerian number* A_c to be $\left\langle y^c \right\rangle_n$, and note that Postnikov [50, Section 16] interprets them as certain mixed volumes up to a normalizing factor, see below.

Observe that $\left\langle y^c \right\rangle_n$ is equal to 0 if $c_n > 0$ because of the presence of the symmetric factor $(x_1 + \cdots + x_n)^{c_n}$ [48, Corollary 3.2]. Hence one may safely restrict one's attention to mixed Eulerian numbers $A_{c_1,\dots,c_n}$ where $c_n = 0$.[1] Henceforth, if we index a mixed Eulerian number by an $(n-1)$-tuple summing to $n-1$, we are implicitly assuming that $c_n = 0$.

The key fact about the mixed Eulerian numbers $A_{(c_1,\dots,c_{n-1})}$ pertinent to our purposes is that they are positive integers. As explained in [50, Section 16], $A_{(c_1,\dots,c_{n-1})}$ equals the mixed volume of the Minkowski sum of hypersimplices $c_1\Delta_{1,n} + \cdots + c_{n-1}\Delta_{n-1,n}$ times $(n-1)!$, which implies positivity. By performing a careful analysis of the volume polynomial $\mathrm{Vol}(P_\lambda)$, Postnikov further provides a combinatorial interpretation for the $A_{(c_1,\dots,c_{n-1})}$ in terms of weighted binary trees; see [50, Theorem 17.7]. A more straightforward combinatorial interpretation for these numbers was provided by Liu [40], in terms of certain permutations with a recursive definition. We omit further details and refer the reader to the articles. Instead we move on to describe some beautiful results due to Petrov [49]. Interestingly, Petrov does not mention mixed Eulerian numbers in his statements, which we believe deserve to be more widely known in this context.

We begin by listing some relations satisfied by the mixed Eulerian numbers that characterize them uniquely.

Lemma 4.1 ([49])**.** *For a fixed positive integer n, the mixed Eulerian numbers $A_{(c_1,\dots,c_n)}$ are completely determined by the following relations:*

(1) $A_{(c_1,\dots,c_n)} = 0$ *if* $c_n > 0$.
(2) $A_{(1^{n-1},0)} = (n-1)!$.
(3) $2A_{(c_1,\dots,c_n)} = A_{(c_1,\dots,c_{i-1}+1,c_i-1,\dots,c_n)} + A_{(c_1,\dots,c_i-1,c_{i+1}+1,\dots,c_n)}$ *if* $i \le n-1$ *and* $c_i \ge 2$.

In the last relation, we interpret c_0 to be c_n.

Proof. Let us sketch Petrov's proof. We have already addressed the first point. The second relation follows immediately by realizing that y^c is a sum of $n!$ monomials when $c = (1^{n-1}, 0)$, and each such monomial contributes 1 upon divided symmetrization; see [48, Section 3.3]. For the third relation we refer the reader to [49, Theorem 4]: it relies on a nice property of divided symmetrization.

The uniqueness follows from the maximum principle: given two solutions to these relations, conside their difference $\delta_{(c_1,\dots,c_n)}$. Assume that δ achieves its maximum value m at $(c_1,\dots,c_n)$: then the third relation implies that m is also achieved at $(c_1,\dots,c_{i-1}+1, c_i-1,\dots,c_n)$ and $(c_1,\dots,c_i-1,c_{i+1}+1,\dots,c_n)$. Applying this argument repeatedly, we can reach all compositions c_{terminal} that have all but one part equal to 1. Since $\delta_{c_{\mathrm{terminal}}} = 0$ by the first two relations, this shows that $\delta = 0$ everywhere. □

Probabilistic interpretation. Petrov turns this characterization into a probabilistic process as follows: Consider $n-1$ coins distributed among the vertices of a regular n-gon, denoted by v_1

[1] The reader comparing our notation to that in [50] should note that Postnikov works under the tacit assumption that $c_n = 0$.

through v_n going cyclically. A *robbing* move consists of picking a vertex v_i that has at least 2 coins, and transferring one coin to either vertex v_{i-1} or v_{i+1} with equal probability. Proceed making such moves until no vertices can be robbed any further. The process terminates almost surely. Note that there are n terminal configurations, each having 1 coin at $n-1$ sites and 0 on the remaining site. Given $c_1, \dots, c_n$ such that $\sum_{1 \le i \le n} c_i = n-1$, let $\mathrm{prob}(c_1, \dots, c_n)$ denote the probability of starting from the initial assignment of c_i coins to v_i and ending in the configuration where v_n has no coins.

Theorem 4.2. ([49, Theorem 5]) *Assuming the notation established earlier, we have that*

$$\mathrm{prob}(c_1, \dots, c_n) = \frac{A_{(c_1,\dots,c_n)}}{(n-1)!}.$$

Petrov arrives at this result by noting that $(n-1)!\mathrm{prob}(c_1, \dots, c_n)$ satisfies the defining relations of the mixed Eulerian numbers listed in Lemma 4.1.

Example 4.3. Suppose $(c_1, c_2, c_3, c_4) = (2, 1, 0, 0)$. It can be checked that $p \coloneqq \mathrm{prob}(c_1, \dots, c_4)$ satisfies $p = \frac{1}{4}(1+p)$ implying that $p = \frac{1}{3}$. This in turn implies that $A_{(2,1,0,0)} = 2$, which is verified easily by expanding $y^{(2,1,0,0)} = x_1^3 + x_1^2 x_2$ and noting that both monomials give 1 upon divided symmetrization.

The preceding probabilistic interpretation renders transparent an interesting relation satisfied by the mixed Eulerian numbers. Define the *cyclic class* of a sequence $c \coloneqq (c_1, \dots, c_n) \in \mathcal{W}_n'$ to be the set of all sequences obtained as cyclic rotations of c. Let us denote this cyclic class by $\mathsf{Cyc}(c)$. It is clear that $|\mathsf{Cyc}(c)| = n$.

Proposition 4.4. ([50, Theorem 16.4], [49, Theorem 4]) *For $c \in \mathcal{W}_n'$, we have that*

$$\sum_{c' \in \mathsf{Cyc}(c)} \frac{A_{c'}}{(n-1)!} = 1.$$

We conclude this section with a discussion on a special class of sequences c. We say that $c \in \mathcal{W}_n'$ is *connected* if c comprises a solitary contiguous block of positive integers and has 0s elsewhere. For instance $(0, 1, 1, 2, 0)$ is connected, whereas $(0, 1, 0, 3, 0)$ is not. Our next result is presented in recent work of Berget, Spink and Tseng [8, Section 7], and was also established independently by the authors.

Proposition 4.5. *Let $\mathbf{a} = (a_1, \dots, a_p)$ be a strong composition of $n-1$. For i, j nonnegative integers let $0^i\mathbf{a}0^j$ denote the sequence obtained by appending i 0s before $\mathbf{a}$ and j 0s after it. Consider the polynomial*

$$\widetilde{A}_{\mathbf{a}}(t) = \sum_{m=0}^{n-p-1} A_{0^m\mathbf{a}0^{n-p-m}} t^m.$$

We have that

$$\sum_{j \ge 0} (1+j)^{a_1}(2+j)^{a_2} \cdots (p+j)^{a_p} t^j = \frac{\widetilde{A}_{\mathbf{a}}(t)}{(1-t)^n}.$$

Example 4.6. Consider $c = (3, 0, 0, 0) \in \mathcal{W}_4'$. Since $\sum_{j \ge 0} (j+1)^3 t^j = \frac{1+4t+t^2}{(1-t)^4}$, Proposition 4.5 tells us that $A_{(3,0,0,0)} = 1$, $A_{(0,3,0,0)} = 4$ and $A_{(0,0,3,0)} = 1$, which are the well-known Eulerian numbers counting the number of permutations in S_3 according to descents.

5. Properties of the numbers a_w

Our starting point in this section is Klyachko's Theorem 3.1, from which we deduce a formula for a_w in terms of mixed Eulerian numbers (Theorem 5.2). From the properties of these mixed Eulerian numbers reviewed in Propositions 4.4 and 4.5, we obtain related properties of a_w in Theorems 5.6 and 5.8 respectively.

5.1. **A positive formula for a_w and first properties.** To start with, we have the following invariance properties of a_w easily deduced from Theorem 3.1:

Proposition 5.1. *For any $w \in \mathrm{S}'_n$, $a_w = a_w^{-1}$ and $a_w = a_{w_o w w_o}$.*

Proof. We have the equality of polynomials $M_w = M_{w^{-1}}$ since $i_1 \dots i_{n-1} \mapsto i_{n-1} \dots i_1$ is a bijection from $\mathrm{Red}(w)$ to $\mathrm{Red}(w^{-1})$, and so we can conclude by Theorem 3.1.

Also, $i_1 \cdots i_{n-1} \mapsto (n-i_1) \cdots (n-i_{n-1})$ is a bijection from $\mathrm{Red}(w)$ to $\mathrm{Red}(w_o w w_o)$, so $M_{w_o w w_o}$ is obtained from M_w after the substitution $x_i \mapsto x_{n-i}$. Because of the symmetry in the presentation of $\mathbf{D}_n$, Theorem 3.1 gives us again that $a_w = a_{w_o w w_o}$. □

The invariance under w_o-conjugation is also a special case of [5, Proposition 3.8], which can be explained geometrically via the duality on $\mathrm{Flag}(n)$. The authors know of no such explanation for the invariance under taking inverses.

We can now state our first formula.

Theorem 5.2. *For any $w \in \mathrm{S}'_n$ and $\mathbf{i} \in \mathrm{Red}(w)$, let $c(\mathbf{i}) = (c_1, \dots, c_{n-1})$ where c_i counts the occurrences of j in $\mathbf{i}$. Then*

$$a_w = \sum_{\mathbf{i} \in \mathrm{Red}(w)} \frac{A_{c(\mathbf{i})}}{(n-1)!}. \tag{5.1}$$

Proof. By Theorem 3.1, it is enough to show that, for any weak composition $c = (c_1, \dots, c_{n-1})$ of $n-1$,

$$\int_{\mathbf{D}_n} u^c = \frac{A_c}{(n-1)!}. \tag{5.2}$$

We now claim that $(n-1)! \int_{\mathbf{D}_n} u^c$ satisfies the three conditions of Lemma 4.1. Indeed the first two are immediate, while the third follows precisely from the relations of $\mathbf{D}_n$. By uniqueness in Lemma 4.1, $(n-1)! \int_{\mathbf{D}_n} u^c = A_c$ as wanted.

Equation (5.2) can also be deduced geometrically from the interpretation of A_c as a normalized mixed volume, cf. [8, 50]. □

Example 5.3. Consider $w = 32415 \in \mathrm{S}'_5$. It has three reduced words $2123, 1213$ and 1231. Given that $A_{2,1,1,0} = 6$ and $A_{1,2,1,0} = 12$, we obtain $a_w = \frac{1}{24}(12 + 6 + 6) = 1$.

The following immediate corollary answers a question asked in [27, Problem 6.6].

Corollary 5.4. *For any $w \in \mathrm{S}'_n$, $a_w > 0$;*

Proof. It follows directly from (5.1) since it expresses a_w as a nonempty sum of positive rational numbers. □

From Section 4 we know also that $A_c \leq (n-1)!$ for any c, so that $a_w \leq |\mathrm{Red}(w)|$ by Theorem 5.2. We will get a quantitative version of the inequality in Theorem 5.6.

Remark 5.5. It is worth remarking that if one considers the computation of $A_{(c_1,\dots,c_{n-1})}$ using its original definition, one has to deal with $\left\langle y_1^{c_1}\dots y_{n-1}^{c_{n-1}}\right\rangle_n$. By repeated applications of Monk's rule [46], we can express $y_1^{c_1}\dots y_{n-1}^{c_{n-1}}$ as a positive integral sum of certain Schubert polynomials in $x_1,\dots,x_{n-1}$. Applying divided symmetrization to the resulting equality results in an expression for $A_{(c_1,\dots,c_{n-1})}$ expressed as a positive integral combination of certain a_w's. It appears nontrivial to 'invert' this procedure and obtain the expression in Theorem 5.2 for the a_w. At any rate, assuming the aforementioned theorem, one does obtain a curious expression for $A_{(c_1,\dots,c_{n-1})}$ in terms of other mixed Eulerian numbers with weights coming from certain chains in the Bruhat order. We omit the details.

Let us also mention that the results of this section have analogues in other types, see Section 8.

5.2. **Indecomposable permutations and sum rules.** In this section we establish two summatory properties of the numbers a_w, based on the notion of factorization of a permutation into indecomposables, which we now recall.

Let $w_1, w_2 \in \mathrm{S}_m \times \mathrm{S}_p$ with $m, p > 0$. The concatenation $w = w_1 \times w_2 \in S_{m+p}$ is defined by $w(i) = w_1(i)$ for $1 \leq i \leq m$ and $w(m+i) = m + w_2(i)$ for $1 \leq i \leq p$. This is an associative operation, sometimes denoted by $\oplus$ and referred to as connected sum. A permutation $w \in \mathrm{S}_n$ is called *indecomposable* if it cannot be written as $w = w_1 \times w_2$ for any $w_1, w_2 \in \mathrm{S}_m \times \mathrm{S}_p$ with $n = m + p$. Note that the unique permutation of $1 \in \mathrm{S}_1$ is indecomposable. The indecomposable permutations for $n \leq 3$ are $1, 21, 231, 312, 321$, and their counting sequence is A003319 in [56]. Permutations can be clearly uniquely factorized into indecomposables: given w in S_n, it has a unique factorization

$$w = w_1 \times w_2 \times \cdots \times w_k, \tag{5.3}$$

where each w_i is an indecomposable permutation in S_{m_i} for certain $m_i > 0$. For instance $w = 53124768 \in \mathrm{S}_8$ is uniquely factorized as $w = 53124 \times 21 \times 1$. We say that w is *quasiindecomposable* if exactly one w_i is different from 1. Thus a quasiindecomposable permutation has the form $1^i \times u \times 1^j$ for u indecomposable $\neq 1$ and integers $i, j \geq 0$.

Given $w \in \mathrm{S}_n$ decomposed as (5.3), its cyclic shifts $w^{(1)},\dots,w^{(k)}$ are given by

$$w^{(i)} = (w_i \times w_{i+1} \cdots \times w_k) \times (w_1 \times \cdots \times w_{i-1}). \tag{5.4}$$

The cyclic shifts of $w = 53124768$, decomposed above, are $w^{(1)} = w = 53124768$, $w^{(2)} = 21386457$ and $w^{(3)} = 16423587$.

These notions are very natural in terms of reduced words: Let the *support* of $w \in \mathrm{S}_n$ be the set of letters in $[n-1]$ that occur in any reduced word for w. Then w is indecomposable if and only if it has full support $[n-1]$. It is quasiindecomposable if its support is an interval in $\mathbb{Z}_{>0}$. Finally, the number k of cyclic shifts of w is equal to n minus the cardinality of the support of w.

Theorem 5.6 (Cyclic Sum Rule)**.** *Let $w \in \mathrm{S}'_n$, and consider its cyclic shifts $w^{(1)},\dots,w^{(k)}$ defined by* (5.3) *and* (5.4)*. We have that*

$$\sum_{i=1}^{k} a_{w^{(i)}} = |\mathrm{Red}(w)|. \tag{5.5}$$

Proof. Let $\mathbf{i} = i_1 \cdots i_{n-1}$ be a reduced word for $w = w^{(1)}$. Consider the words $\mathbf{i}[t] = (i_1 + t) \cdots (i_{n-1} + t)$ for $t = 0, \ldots, n-1$, where the values $i_j + t$ are considered as their residues modulo n with representatives belonging to the interval $\{1, \ldots, n\}$. Let $0 = t_1 < \cdots < t_k$ be the values of t for which n does *not* occur in $\mathbf{i}[t]$. Then in the notation of (5.4), we have $t_j = \sum_{i=1}^{j-1} m_i$. Moreover, $\mathbf{i} \mapsto \mathbf{i}[t_j]$ is a bijection between $\mathrm{Red}(w)$ and $\mathrm{Red}(w^{(j)})$ for any j.

Fix $\mathbf{i} = i_1 \cdots i_{n-1} \in \mathrm{Red}(w)$, and $c = (c_1, \ldots, c_n) \in \mathcal{W}'_n$ where c_i is the number of occurrences of i in $\mathbf{i}$. For the reduced word $\mathbf{i}[t_j]$, the corresponding vector is given by the cyclic shift $c[j] = (c_{t_j+1}, \ldots, c_n, c_1, \ldots, c_{t_j})$. By definition of the indices t_j, the $c[j]$ are exactly the cyclic shifts of c that have a nonzero last coordinate. By Proposition 4.4, we have

$$\sum_{j=0}^{k-1} \frac{A_{c[j]}}{(n-1)!} = 1.$$

If we sum the previous identity over all reduced words of w, then by Theorem 5.2 applied to each term of the previous sum, we obtain (5.5). $\square$

Example 5.7. Let $w = 53124768 \in \mathrm{S}'_8$ already considered earlier. Then one has $|\mathrm{Red}(w)| = 63$ while $a_{w^{(1)}} + a_{w^{(2)}} + a_{w^{(3)}} = 6 + 21 + 36 = 63$.

We now present a refined property of the numbers a_w when w is quasiindecomposable, giving a simple way to compute them in terms of principal specializations of Schubert polynomials. Given a permutation u of length ℓ and $m \geq 0$, consider

$$\nu_u(m) := \nu_{1^m \times u} = \mathfrak{S}_{1^m \times u}(1, 1, \ldots). \tag{5.6}$$

By Macdonald's identity (2.15) we have

$$\nu_u(m) = \frac{1}{\ell!} \sum_{\mathbf{i} \in \mathrm{Red}(u)} (i_1 + m)(i_2 + m) \cdots (i_\ell + m), \tag{5.7}$$

which is a *polynomial in* m of degree ℓ. Therefore (see [58] for instance) there exist integers $h^u_m \in \mathbb{Z}$ for $m = 0, \ldots, \ell$ such that

$$\sum_{j \geq 0} \nu_u(j) t^j = \frac{\sum_{m=0}^{\ell} h^u_m t^m}{(1-t)^{\ell+1}}. \tag{5.8}$$

Moreover, the numbers h^u_m are known to sum to $\ell!$ times the leading term of $\nu_u(m)$, that is $\sum_{m=0}^{\ell} h^u_m = |\mathrm{Red}(u)|$. Thus the following theorem is a refinement of Theorem 5.6 in the case of quasiindecomposable permutations.

Theorem 5.8. *Assume that $u \in \mathrm{S}_{p+1}$ is* indecomposable *of length $n-1$. Define quasiindecomposable permutations $u^{[m]} \in \mathrm{S}'_n$ for $m = 0, \ldots, n-p-1$ by $u^{[m]} := 1^m \times u \times 1^{n-p-1-m}$. Then*

$$h^u_m = \begin{cases} a_{u^{[m]}} & \text{if } m < n-p; \\ 0 & \text{if } m \geq n-p \end{cases}$$

Equivalently, one has

$$\sum_{j \geq 0} \nu_u(j) t^j = \frac{\sum_{m=0}^{n-p-1} a_{u^{[m]}} t^m}{(1-t)^n}. \tag{5.9}$$

Proof. The map $\rho_m : i_1 \cdots i_{n-1} \mapsto (i_1 + m) \cdots (i_{n-1} + m)$ is a bijection between $\mathrm{Red}(u)$ and $\mathrm{Red}(u^{[m]})$ for $m = 0, \ldots, n-p-1$.

Fix $\mathbf{i} = i_1 \cdots i_{n-1} \in \mathrm{Red}(u)$. Since u is indecomposable, it has full support, so that $c(\mathbf{i})$ has the form $(a_1, \ldots, a_p, 0, 0, \ldots)$ where $\mathbf{a} = (a_1, \ldots, a_p) \vDash n-1$. Then $0^m\mathbf{a}$ is equal to $c(\rho_m(\mathbf{i}))$ for $m = 0, \ldots, n-p-1$. We can apply Proposition 4.5 to $\mathbf{a}$, and we get:

$$\sum_{j\geq 0} (1+j)^{a_1}(2+j)^{a_2}\cdots(p+j)^{a_p}t^j = \frac{\sum_{m=0}^{n-p-1} A_{c(\rho_m(\mathbf{i}))}t^m}{(1-t)^n}.$$

We now sum this last identity over all $\mathbf{i} \in \mathrm{Red}(u)$. On the left hand side, for a fixed j, the coefficients sum to $(n-1)!\nu_w(j)$ by Macdonald's identity (2.15). On the right hand side, for a fixed m the coefficients $A_{c(\rho_m(\mathbf{i}))}$ sum to $(n-1)!a_{u^{[m]}}$ by Theorem 5.2. This completes the proof of (5.9). ☐

Example 5.9. Consider $n = 7$ and $u = 4321 \in S_4$ an indecomposable permutation. We have that $u^{[0]} = 4321567$, $u^{[1]} = 1543267$, $u^{[2]} = 1265437$, and $u^{[3]} = 1237654$. It is easily checked that

$$\sum_{j\geq 0} \nu_u(j)t^j = \frac{1+7t+7t^2+t^3}{(1-t)^7}$$

Take particular note of the fact that coefficients in the numerator on the right hand side are all positive, which is a priori not immediate. Theorem 5.8 then tells us that $a_{u^{[0]}} = 1$, $a_{u^{[1]}} = 7$, $a_{u^{[2]}} = 7$, and $a_{u^{[3]}} = 1$. Section 7 offers a complete explanation for why these numbers arise.

Observe that by extracting coefficients, Theorem 5.8 gives a *signed* formula for a_w for any quasiindecomposable w in terms of principal specializations of shifted Schubert polynomials: for any $u \in \mathrm{S}_{p+1}$ indecomposable of length $n-1$, and $m = 0, \ldots, n-p-1$, we have that

$$a_{u^{[m]}} = \sum_{j=0}^{n} \nu_u(j)(-1)^{m-j}\binom{n}{m-j}. \tag{5.10}$$

A last observation is that the stability properties from Proposition 5.1 are nicely reflected in Theorem 5.8. The fact that $a_w = a_{w^{-1}}$ for any w quasiindecomposable is immediate since $\nu_u(j) = \nu_{u^{-1}}(j)$ for any j by (5.7), so that the right hand side of (5.8) for u and u^{-1} coincide.

The stability under w_o-conjugation is more interesting: let $\bar{u} = w_o^{p+1}uw_o^{p+1}$ where w_0^{p+1} denotes the longest word in S_{p+1}. Using [58, 4.2.3]) we deduce from (5.9) that

$$\sum_{j\geq 1} \nu_u(-j)t^j = (-1)^{n-1}\frac{\sum_{m=0}^{n-p-1} a_{u^{[m]}}t^{n-m}}{(1-t)^n}$$

Now $\nu_u(-i) = 0$ for $i = 1, \ldots, p$ since u has full support, so, using the change of variables $j \mapsto j-p-1$, we can rewrite the previous equation as

$$\sum_{j\geq 0} \nu_u(-j-p-1)t^j = (-1)^{n-1}\frac{\sum_{m=0}^{n-p-1} a_{u^{[m]}}t^{n-m-p-1}}{(1-t)^n}$$

We also have $\nu_{\bar{u}}(j) = (-1)^{n-1}\nu_u(-j-p-1)$ easily from (5.7). Putting these together, we get $a_{\bar{u}^{[m]}} = a_{u^{[n-1-p-m]}}$ for any $m \leq n-p-1$. This is equivalent to the fact that $a_w = a_{w_oww_o}$ for any $w \in \mathrm{S}'_n$ quasiindecomposable.

6. Combinatorial interpretation of a_w in special cases

We identify certain special classes of permutations for which we have a combinatorial interpretation. Assume $n \geq 2$ throughout this section.

6.1. **Łukasiewicz permutations.**

Definition 6.1. A weak composition $(c_1, \ldots, c_n) \in \mathcal{W}_n'$ is called *Łukasiewicz* if it satisfies $c_1 + \cdots + c_k \geq k$ for any $k \in \{1, \cdots, n-1\}$.

A permutation $w \in \mathrm{S}_n'$ is *Łukasiewicz* if code(w) is a Łukasiewicz composition.

We note that $c_1 + \cdots + c_n = n - 1$ since c is assumed to be in $\mathcal{W}_n'$, so that the inequality in Definition 6.1 fails for $k = n$. Let $\mathcal{LP}_n$ be the set of Łukasiewicz permutations and $\mathcal{LC}_n$ the set of Łukasiewicz compositions. If $Y = \{y_0, y_1, \ldots\}$ is an alphabet, then the words $y_{c_1} y_{c_2} \cdots y_{c_n}$ for $c \in \mathcal{LC}_n$ are known as *Łukasiewicz words* in Y [41]. These are known to be counted by Catalan numbers $Cat_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}$.

Example 6.2. There are 5 compositions in $\mathcal{LC}_4$:

$$(3,0,0,0), (2,1,0,0), (2,0,1,0), (1,2,0,0), (1,1,1,0)$$

corresponding to the Łukasiewicz permutations $4123, 3214, 3142, 2413, 2341$.

Proposition 6.3. *For $n \geq 1$, we have $|\mathcal{LP}_n| = |\mathcal{LC}_n| = Cat_{n-1}$.*

Proof. We have already argued above that $|\mathcal{LC}_n| = Cat_{n-1}$. If $c \in \mathcal{LC}_n$ then $c_i \leq n - i$ for all i since

$$c_i \leq c_i + \ldots + c_n = n - 1 - (c_1 + \cdots + c_{i-1}) \leq n - 1 - (i-1) = n - i.$$

It follows that the code is a bijection from $\mathcal{LP}_n$ to $\mathcal{LC}_n$. □

Our next proposition states that the set of Łukasiewicz permutations is stable under taking inverses.

Proposition 6.4. *If $w \in \mathcal{LP}_n$ then $w^{-1} \in \mathcal{LP}_n$.*

This claim is a priori not clear from the definition, because determining code(w^{-1}) from code(w) is a convoluted process. We give a proof based on an alternative characterization of $\mathcal{LP}_n$ in the appendix.

6.2. **Computation of a_w for Łukasiewicz permutations.**

We recall Postnikov's result [50] (see also [48, 49]) for the evaluation of divided symmetrization on monomials. Let $c = (c_1, \ldots, c_n) \in \mathcal{W}_n'$. Define the subset $S_c \subseteq [n-1]$ by $S_c := \{k \in [n-1] \mid \sum_{i=1}^{k} c_i < k\}$. Then

$$\left\langle x_1^{c_1} \cdots x_n^{c_n} \right\rangle_n = (-1)^{|S_c|} \beta_n(S_c), \tag{6.1}$$

Here $\beta_n(S)$ is the number of permutations in S_n with descent set S as defined in Section 2.1. Recall that we have $a_w = \left\langle \mathfrak{S}_w \right\rangle_n$, see (3.2), so that by applying (6.1) to each monomial in the pipe dream expansion (2.13) of $\mathfrak{S}_w$, we obtain the formula:

$$a_w = \sum_{\gamma \in \mathrm{PD}(w)} (-1)^{|S_{c(\gamma)}|} \beta_n(S_{c(\gamma)}). \tag{6.2}$$

In general, this signed sum seems hard to analyze and simplify, and positivity is far from obvious. The nice case where this approach works corresponds precisely to $w \in \mathcal{LP}_n$.

Theorem 6.5. *If $w \in \mathcal{LP}_n$, then $a_w = |\mathrm{PD}(w)|$.*

Proof. We examine the expansion (2.13) into pipe dreams. If a pipe dream γ has weight $(c_1, \dots, c_n)$, then a ladder move transforms it into a pipe dream γ' with weight $(c'_1, \dots, c'_n)$ where $c'_i = c'_i + 1$, $c'_j = c'_j - 1$ for some $i < j$ while $c'_k = c_k$ for $k \neq i, j$. In particular $(c_1, \dots, c_n) \in \mathcal{LC}_n$ implies $(c'_1, \dots, c'_n) \in \mathcal{LC}_n$.

By definition the bottom pipe dream γ_w has weight code(w) for any w. Assume $w \in \mathcal{LP}_n$ so that the weight of $\mathcal{LC}_n$ is in $\mathcal{LC}_n$. It then follows from Theorem 2.3 that all pipe dreams in the expansion (2.13) have weight in $\mathcal{LC}_n$.

If $(c_1, \dots, c_n) \in \mathcal{LC}_n$ then $S_c = \emptyset$ and so $\left\langle x_1^{c_1} \cdots x_n^{c_n} \right\rangle_n = 1$ because $\beta_n(S_c)$ contains only the identity of S_n. Putting things together, we have for any $w \in \mathcal{LP}_n$,

$$a_w = \left\langle \mathfrak{S}_w \right\rangle_n = \sum_{\gamma \in \mathrm{PD}(w)} \left\langle x^\gamma \right\rangle_n = |\mathrm{PD}(w)| = \nu_w,$$

which concludes the proof. $\square$

Example 6.6. Let $w = 31524 \in \mathcal{LP}_5$ with code $(2, 0, 2, 0, 0)$. $\mathrm{PD}(w)$ consists of 4 elements, and thus by Theorem 6.5 we get $a_w = 4$.

The combinatorial interpretation $a_w = |\mathrm{PD}(w)|$ shows $a_w > 0$ since $\mathrm{PD}(w)$ contains at least the bottom pipe dream. By Proposition 6.4, $\mathcal{LP}_n$ is stable under inverses, and so the stability under taking inverses from 5.1 is equivalent in this case to $|\mathrm{PD}(w)| = |\mathrm{PD}(w^{-1})|$. This follows combinatorially from the transposition of pipe dreams along the diagonal.

Note that $\mathcal{LP}_n$ is not stable under conjugation by w_o: for instance, for the permutation 3214 in $\mathcal{LP}_4$ we have $w_o^4(3214)w_o^4 = 1432 \notin \mathcal{LP}_4$. Thanks to Proposition 5.1, we have

Corollary 6.7. *$a_w = \nu_{w_o w w_o}$ if $w_o w w_o \in \mathcal{LP}_n$.*

So for instance we get $a_{1432} = \nu_{3214} = 1$. Notice that this is different from $\nu_{1432} = 5$.

Remark 6.8. The cardinality $|\mathcal{LP}_n| = \frac{1}{n}\binom{2n-2}{n-1}$ is asymptotically equal to $4^{n-1} n^{-3/2}/\sqrt{\pi}$ by Stirling's formula. Compared to the asymptotics for $|\mathrm{S}'_n|$ computed in [45], one sees that the ratio $|\mathcal{LP}_n|/|\mathrm{S}'_n|$ is asymptotically equivalent to C/n for an explicit constant C.

Remark 6.9. A *dominant* permutation is defined as a permutation whose code is a partition, or equivalently as a 132-avoiding permutation [44]. Such a permutation has a single pipe dream (necessarily its bottom pipe dream), and so $a_w = 1$ by Theorem 6.5 for any $w \in \mathrm{S}'_n$. By the invariance under w_o-conjugation (Corollary 6.7) 213-avoiding permutations w in S'_n also satisfy $a_w = 1$. Up to $n = 11$ these are the only classes of permutations for which a_w is equal to 1.

We now connect Łukasiewicz permutations with the cyclic shifts of permutations.

Proposition 6.10. *For $w \in \mathrm{S}'_n$, the permutations $w^{(i)}$ are pairwise distinct, and exactly one of them is Łukasiewicz.*

Proof. Denote by $(c_1, \dots, c_n)$ the code of w. The *cycle lemma* [42, Lemma 9.1.10] says that all shifts $(c_j, c_j + 1, \dots, c_n, c_1, \dots, c_{j-1})$ are distinct, and exactly one of them is in $\mathcal{LC}_n$. Now these shifts are codes of permutations in S'_n exactly for the permutations $w^{(i)}$, which completes the proof. $\square$

Notice that as a consequence of Theorems 6.5 and 5.6, we also have the following corollary.

Corollary 6.11. *If $w \in \mathcal{LP}_n$, then* $|\mathrm{PD}(w)| \leq |\mathrm{Red}(w)|$.

It would be interesting to find a combinatorial proof of this corollary, for instance by finding an explicit injection from $\mathrm{PD}(w)$ to $\mathrm{Red}(w)$.

6.3. **Coxeter elements.** This case is a subcase of the previous one with particularly nice combinatorics. A *Coxeter element* of S_n is a permutation that can be written as the product of all elements of the set $\{s_1, s_2, \ldots, s_{n-1}\}$ in a certain order. Let Cox_n be the set of all Coxeter elements of S_n. Since the defining expressions for Coxeter elements are clearly reduced, we have $\mathrm{Cox}_n \subseteq \mathrm{S}'_n$.

Coxeter elements are naturally indexed by subsets of $[n-2]$ as follows: for w a Coxeter element, define $I_w \subset [n-2]$ by the following rule: $i \in S_w$ if and only if i occurs before $i+1$ in a reduced word for w (equivalently, in *all* reduced words for w). Conversely any subset of $[n-2]$ determines a unique Coxeter element, and therefore we have $|\mathrm{Cox}_n| = 2^{n-2}$.

Lemma 6.12. $\mathrm{Cox}_n \subseteq \mathcal{LP}_n$.

Proof. We do this by characterizing codes of Coxeter elements. Let $w \in \mathrm{Cox}_n$, and $I_w = \{i_1 < \ldots < i_k\} \subset [n-2]$ as defined above. To I_w corresponds $\alpha_w = (i_1, i_2 - i_1, \ldots, i_k - i_{k-1}, n-1-i_k)$ a composition of $n-1$ using a folklore bijection between subsets and compositions. Finally, writing $\alpha_w = (\alpha_1, \ldots, \alpha_{k+1}) \vDash n-1$, define the weak composition c_w of $n-1$ with n parts by inserting $\alpha_i - 1$ zeros after each α_i, and append an extra zero at the end. We claim that $c_w = \mathrm{code}(w)$, leaving the easy verification to the reader.

To illustrate this result, pick $w = 2513746 \in \mathrm{Cox}_7$, with $431265 \in \mathrm{Red}(w)$. We compute successively $I_w = \{1,4\} \subset [5]$, $\alpha_w = (1,3,2) \vDash 6$ and finally $c_w = (1,3,0,0,2,0,0)$ which is indeed the code of w.

An alternative proof is to use Proposition A.2 here: using pipe dreams it is easily shown that $\bar{a}(w) = (1,1,\ldots,1,0)$ for any Coxeter element, and this in fact characterizes such elements. Since $(1,1,\ldots,1,0) \in \mathcal{LC}_n$, Proposition A.2 ensures that Coxeter elements belong to $\mathcal{LP}_n$. □

It follows that $a_w = |\mathrm{PD}(w)|$ if $w \in \mathrm{Cox}_n$ by Theorem 6.5. We note that Sean Griffin [25] has managed to give a geometric proof of this fact using Gröbner degeneration techniques.

Proposition 6.13. *If $w \in \mathrm{Cox}_n$, then $a_w = \beta_{n-1}(I_w)$.*

Proof. It is enough to exhibit a bijection ϕ between $\mathrm{PD}(w)$ and permutations of S_{n-1} with descent set I_w. If $n = 2$ then $w = s_1$ and we associate to it the identity permutation in S_1. Now let $w \in \mathrm{Cox}_{n+1}$ for $n \geq 2$. Note that $\gamma \in \mathrm{PD}(w)$ has exactly one $+$ in each antidiagonal A_k given by $i + j = k - 1$ for $k = 1, \ldots, n$; we label them $+_1, \ldots, +_n$. Removing $+_n$ gives a pipe dream γ' in $\mathrm{PD}(w')$ for a Coxeter element $w' \in \mathrm{Cox}_n$. By induction we can assume that we have constructed $\sigma' = \phi(\gamma') \in \mathrm{S}_{n-1}$ with descent set $I_{w'} \subset [n-2]$.

Let i, j be the rows in γ containing $+_{n-1}, +_n$ respectively.. Then define σ by incrementing by 1 all values in σ' larger or equal to $n+1-j$, and inserting $n+1-j$ at the end of σ'. By immediate induction σ' is a permutation ending with $n+1-i$, and $\mathrm{Des}(\sigma') = I_{w'}$. Noting that $I_w = I_{w'} \cup \{n-1\}$ if $j > i$ and $I_w = I_{w'}$ if $j \leq i$, one sees that $\mathrm{Des}(\sigma) = I_w$. We leave the verification that this is a bijection to the reader. □

As interesting special cases, consider the Coxeter elements w_{odd}, resp. w_{even}, of S_n defined by the fact that by $I_{w_{odd}}$, resp. $I_{w_{even}}$, consists of all odd, resp. even, integers in $[n-2]$. Then the number $\beta_{n-1}(I_{w_{odd}}) = \beta_{n-1}(I_{w_{even}})$ is the *Euler number* E_{n-1} which by definition counts the number of *alternating permutations* in S_{n-1}. Data up to $n = 11$ indicates that the value $a_{w_{odd}} = a_{w_{even}} = E_n$ is the maximal value of a_w over S'_n, and is obtained for these two permutations precisely.

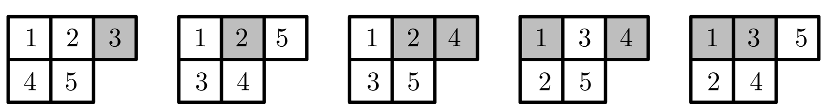
Remark 6.14. Theorem 3.1 can alternatively be applied directly here to give $a_w = |\mathrm{Red}(w)|$ instead, since all terms in the sum contribute 1. The statement of Proposition 6.13 can be deduced from this evaluation also, since reduced words of Coxeter elements are naturally in one-to-one correspondence with standard tableaux of a certain *ribbon shape* attached to w, themselves naturally in bijection with permutations having descent set I_w. We skip the details.

6.4. **Grassmannian permutations.** In this section we give a combinatorial interpretation of a_w when w is a Grassmannian permutation (Theorem 6.16). We remind the reader again that Klyachko [37] considers this case as well, but obtains a signed expression for a_w. Note that this case will be extended to the much larger class of vexillary permutations in Section 7.

Definition 6.15. A permutation in S_∞ is *Grassmannian* if it has a unique descent. It is m-Grassmannian if this unique descent is $m \geq 1$.

The codes $(c_1, c_2 \ldots)$ of m-Grassmannian permutations are characterized by $0 \leq c_1 \leq c_2 \leq \cdots \leq c_m$ (with $c_m > 0$) while $c_i = 0$ for $i > m$. A Grassmannian permutation $w \in \mathrm{S}_\infty$ is thus encoded by the data $(m, \lambda(w))$, which must satisfy $m \geq \ell(\lambda(w))$. Conversely any m, λ that satisfy $m \geq \ell(\lambda)$ corresponds to a permutation in S_∞. Moreover, such a permutation is in S_n if and only if $n \geq m + \lambda_1$.

Recall that a *standard Young tableau* T of shape $\lambda \vdash n$ is a filling of the Young diagram of λ by the integers $\{1, \ldots, n\}$ that is increasing along rows and columns. A *descent* of T is an integer $i < n$ such that $i + 1$ occurs in a row strictly below i (here we assume the Young diagram uses the English notation, with weakly decreasing rows from top to bottom). As illustrated below, for the shape $(3, 2)$ for which there are 5 tableaux, the cells containing descents are shaded.



Let $\mathrm{SYT}(\lambda)$ be the set of standard Young tableaux of shape λ and $\mathrm{SYT}(\lambda, d)$ be the subset thereof containing tableaux with exactly d descents.

Theorem 6.16. *Let $w \in \mathrm{S}'_n$ be a Grassmannian permutation with descent m and shape λ. Then a_w is equal to* $\mathrm{SYT}(\lambda, m-1)$.

Proof. In this case, the Schubert polynomial $\mathfrak{S}_w$ is known to be the Schur polynomial $s_\lambda(x_1, \cdots, x_m)$ [44, Proposition 2.6.8]. We thus have to compute $a_w = \left\langle s_\lambda(x_1, \ldots, x_m) \right\rangle_n$. This is a consequence of the results of [48] about divided symmetrizations of (quasi)symmetric functions: see Proposition 4.4 and Example 4.6 in [48]. $\square$

Example 6.17. Consider the permutations $w_1 = 351246$ and $w_2 = 146235$, which are the two Grassmannian permutations in S'_6 with shape $(3, 2)$. Note that w_1 has descent 2 while w_2 has descent 3. So $a_{w_1} = \mathrm{SYT}(\lambda, 1) = 2$ and $a_{w_2} = \mathrm{SYT}(\lambda, 1) = 3$ from the inspection above.

It is interesting to deduce $a_w > 0$ and the invariance under w_o-conjugation (cf. Section 5.1) from this combinatorial interpretation. Note that the inverse of a Grassmannian permutation is not in general Grassmannian, so at this stage the invariance under inverses is not apparent.

Positivity of a_w for w Grassmannian can be shown to be equivalent to the following statement: *for any shape λ and any integer d satisfying $\lambda'_1 - 1 \leq d \leq |\lambda| - \lambda_1$, then* $\mathrm{SYT}(\lambda, d) \neq \emptyset$. It is indeed possible to construct explicitly such a tableau in $\mathrm{SYT}(\lambda, d)$; we omit the details.

Now suppose w is m-Grassmannian with shape $\lambda \vdash n-1$. Then $w_o w w_o$ is also Grassmannian, with descent $n-m$ and associated shape λ', the transpose of λ. It is then a simple exercise to show that transposition implies $\mathrm{SYT}(\lambda, m-1) = \mathrm{SYT}(\lambda', n-m-1)$.

We finish by giving a pleasant evaluation for a family of mixed Eulerian numbers. Recall that the *content* of a cell in the ith row and jth column in the Young diagram of a partition λ is defined to be $j - i$.

Corollary 6.18. *Let $w \in \mathrm{S}'_n$ be an m-Grassmannian permutation of shape $\lambda \vdash n-1$. For $i = 1, \ldots, n-1$, let c_i be the number of cells of λ with content $m-i$. Then*

$$A_{c_1,\ldots,c_{n-1}} = |\mathrm{SYT}(\lambda, m-1)| \prod_{(i,j)\in\lambda} h(i,j),$$

where $h(i,j) = \lambda_i + \lambda'_j - i - j + 1$ is the hook-length of the cell (i,j) in λ.

Proof. Grassmannian permutations are fully commutative as they are 321-avoiding, so all their reduced expressions have the same value for $c(\mathbf{i})$. It follows from Theorem 3.1 that

$$a_w = \frac{|\mathrm{Red}(w)|}{(n-1)!} A_{c_1,\ldots,c_{n-1}}.$$

Now

$$|\mathrm{Red}(w)| = |\mathrm{SYT}(\lambda)| = \frac{(n-1)!}{\prod_{(i,j)\in\lambda} h(i,j)}$$

by the hook-length formula. The conclusion follows from Theorem 6.16. □

We discuss the *fully commutative* case in Section 8.

7. The case of vexillary permutations

In this section we will give a combinatorial interpretation to a_w for w vexillary in S'_n.

Definition 7.1. A permutation is *vexillary* it it avoids the pattern 2143.

They were introduced in [39]. This is an important class of permutations in relation to Schubert calculus, containing both dominant and Grassmannian permutations. The Stanley symmetric function F_w [57] is equal to a single Schur function if and only if w is vexillary. Combinatorially, vexillary permutations correspond to leaves of the Lascoux-Schützenberger tree, and play a special role in the Edelman-Greene; see [44] and the references therein.

Proposition 7.2. *The class of vexillary permutations in S_n is closed under taking inverses, and conjugation by w_o. Moreover, vexillary permutations are quasiindecomposable.*

Proof. Closure under inverses, resp. conjugation by w_o, follows immediately from the fact that the pattern 2143 is an involution, resp. is invariant under conjugation by w_o.

Now suppose $w \in \mathrm{S}_n$ is *not* quasiindecomposable. Then there exist indecomposable $w_i, w_j \neq 1$ with $i < j$ in the factorization (5.3). There exists an inversion in each of w_i, w_j, and any pair of such inversions give an occurrence of the pattern 2143 in w, so that w is not vexillary. □

In particular we will be able to use Theorem 5.8. We first need to recall certain tableau combinatorics related to vexillary permutations. Then we shall relate these tableaux to a certain model of *ϵ-tableaux*, in order to apply the theory of (P, ω)-partitions to interpret the left hand side of (5.9) in the vexillary case, and ultimately identify the combinatorial interpretation for a_w.

7.1. **Flagged tableaux for vexillary permutations.** It is known, see [39, 61], that the Schubert polynomials of vexillary permutations are *flagged Schur functions*, which we now describe.

Fix a partition λ with l parts, and let $b = (b_1, \dots, b_l)$ be a nondecreasing sequence of positive integers $1 \leq b_1 \leq \dots \leq b_l$. A *flagged tableau* T of shape λ and flag b is a semistandard Young tableau of shape λ such that entries in the ith row of T lie in $[b_i]$. The weight $\mathbf{x}^T$ of T is the monomial $x_1^{m_1} x_2^{m_2} \cdots$ with m_i the number of entries i in T. Let $\mathrm{SSYT}(\lambda; b)$ be the set of flagged tableaux of shape λ and flag $\mathbf{b}$. Then

$$s_\lambda(\mathbf{x}; b) = \sum_{T \in \mathrm{SSYT}(\lambda; \mathbf{b})} \mathbf{x}^T$$

is the corresponding *flagged Schur function*.

Now let $w \in \mathrm{S}_\infty$ be a partition with code $c = \mathrm{code}(w)$. Recall that the shape $\lambda(w)$ is the partition obtained by sorting the nonzero entries of c in nonincreasing order. Given i such that $c_i > 0$, define e_i to be the maximal j such that $c_j \geq c_i$. The *flag* $\phi(w)$ of w is defined by ordering the e_i in nondecreasing order.

This can be expressed in a more compact way as follows: Write λ uniquely in the form $\lambda = (p_1^{m_1}, p_2^{m_2}, \dots, p_r^{m_r})$ with $p_1 > p_2 > \dots > p_r$. For $1 \leq q \leq r$, let ϕ_q be the maximum index j such that $c_j \geq p_q$. Then it is clear that $\phi(w) = (\phi_1^{m_1}, \dots, \phi_r^{m_r})$.

Example 7.3. Consider $w = 812697354 \in S_9$. We have $\mathrm{code}(w) = (7, 0, 0, 3, 4, 3, 0, 1, 0)$. We compute $e_1 = 1$, $e_4 = 6$, $e_5 = 5$, $e_6 = 6$ and $e_8 = 8$. Thus $\phi(w) = (1, 5, 6, 6, 8)$.

Alternatively, express $\lambda(w) = (7, 4, 3^2, 1)$. We have $\phi_1 = 1$, $\phi_2 = 5$, $\phi_3 = 6$, and $\phi_4 = 8$. This gives the same flag as before.

We note further that an m-Grassmannian permutation has flag $\phi = (m, \dots, m)$, while a dominant permutation has flag $\phi = (m_1^{m_1}, (m_1 + m_2)^{m_2}, \dots, (m_1 + m_2 + \dots + m_r)^{m_r})$.

If w is vexillary of shape $\lambda(w)$, then $\mathfrak{S}_w = s_{\lambda(w)}(\mathbf{x}, \phi(w))$ (cf. [39, 61]) and in particular

$$\nu_w = |\mathrm{SSYT}(\lambda(w), \phi(w))|.$$

Proposition 7.4. [39, 43] *A vexillary permutation is characterized by the data of its shape and flag. Moreover,* $(\lambda = (p_1^{m_1}, \dots, p_r^{m_r}), \phi = (\phi_1^{m_1}, \dots, \phi_r^{m_r}))$ *is equal to* $(\lambda(w), \phi(w))$ *for w vexillary if and only if the following inequalities are satisfied:*

$$\phi_q \geq m_1 + \dots + m_q \qquad \textit{for } q = 1, \dots, r; \tag{7.1}$$

$$0 \leq \phi_{q+1} - \phi_q \leq m_{q+1} + p_q - p_{q+1} \qquad \textit{for } q = 1, \dots, r-1. \tag{7.2}$$

The first set of inequalities is easy to prove (and valid for any permutation). The second one is more involved, cf. [43]. It is interesting to consider the extreme cases of each:

- $\phi_q = m_1 + \cdots + m_q$ for $q = 1, \ldots, r$ iff w is dominant.
- $\phi_q = \phi_{q+1}$ for $q = 1, \ldots, r-1$ iff w is Grassmannian.
- $\phi_{q+1} - \phi_q = m_{q+1} + p_q - p_{q+1}$ for $q = 1, \ldots, r-1$ iff w is inverse Grassmannian, that is w^{-1} is Grassmannian.

7.2. **Plane partitions with arbitrary strict conditions on rows and columns.** We fix $\lambda = (\lambda_1, \ldots, \lambda_l)$, where $l = \lambda'_1$ is the number of parts. Recall that a plane partition of shape λ is an assignment $T_{i,j} \in \{0, 1, 2, \ldots\}$ for $(i, j) \in \lambda$ that is weakly decreasing along rows and columns. In other words, if P_λ is the poset of cells of λ in which $c \leq c'$ if c is to the northwest of c', then a plane partition of shape λ is a P_λ-partition in the sense of Stanley [58, Section 4.5].

Definition 7.5. A *signature* for λ is an ordered pair $\epsilon = (e, f) \in \{0,1\}^{l-1} \times \in \{0,1\}^{\lambda_1 - 1}$.

An *ϵ-partition* of shape λ is a plane partition $(T_{i,j})$ of shape λ such that $T_{i,j} > T_{i+1,j}$ if $e_i = 1$ and $T_{i,j} > T_{i,j+1}$ if $f_j = 1$.

Thus, in an ϵ-partition entries must strictly decrease between rows (*resp.* columns) i and $i+1$ if $e_i = 1$ (*resp.* $f_i = 1$). Let $\Omega(\lambda, \epsilon, N)$ be the number of ϵ-partitions of shape λ with maximal entry at most N. An example of ϵ-partition is given in Figure 2 for $N = 6$. Plane partitions correspond to the signature $e_i = f_j = 0$ for all i and j.

A *labeling* ω of P_λ is a bijection from P_λ to $\{1, \ldots, |\lambda|\}$. Let ω_ϵ be a *compatible labeling*: that is, it satisfies $\omega_\epsilon(i,j) > \omega_\epsilon(i+1,j)$ if and only if $e_i = 1$, and $\omega_\epsilon(i,j) > \omega_\epsilon(i,j+1)$ if and only if $f_j = 1$. Such a labeling always exists: indeed, let $G_{\lambda,\epsilon}$ be the directed graph whose underlying undirected graph is the Hasse diagram of P_λ, and with orientation given by $(i,j) \to (i,j+1)$ if and only if $e_i = 1$, and $(i,j) \to (i+1,j)$ if and only if $f_j = 1$. The orientation is easily seen to be *acyclic*, which ensures the existence of compatible labelings ω_ϵ since those are precisely the *topological orderings* of $G_{\lambda,\epsilon}$, that is the linear orderings of its vertices such that if $u \to v$ then $\omega_\epsilon(u) < \omega_\epsilon(v)$. These exist exactly when the graph is a directed acyclic graph (DAG).

f: 0 1 0 0 1 0

e: 0 1 0 0

6	6	5	5	5	2	2
6	6	4	3	3	2	0
5	4	3	2	2	1	
5	4	2				
3	3	2				

f: 0 1 0 0 1 0

e: 0 1 0 0

1	3	2	4	6	5	7
14	16	15	17	19	18	20
8	10	9	11	13	12	
21	23	22				
24	26	25				

FIGURE 2. $\lambda = (7,7,6,3,3)$ with signature $\epsilon = (0100, 010010)$. An ϵ-partition (left) and a compatible labeling ω_ϵ (right).

We now recognize that an ϵ-partition of shape λ is precisely a $(P_\lambda, \omega_\epsilon)$-partition [59, Section 7.19]. By the general theory of (P, ω)-partitions, we get the following result: Let $SYT(\lambda)$ be the set of standard tableaux of shape λ. An ω_ϵ-*descent* of $T \in \mathrm{SYT}(\lambda)$ is an entry $k < |\lambda|$ such that $\omega_\epsilon(T^{-1}(k)) > \omega_\epsilon(T^{-1}(k+1))$. Let $\mathrm{des}(T; w_\epsilon)$ be the number of ω_ϵ-descents of T. Then

$$\sum_{N\geq 0} \Omega(\lambda,\epsilon,N)t^N = \frac{\sum_{T\in \mathrm{SYT}(\lambda)} t^{\mathrm{des}(T;w_\epsilon)}}{(1-t)^{|\lambda|+1}}. \tag{7.3}$$

7.3. **From ϵ-tableaux to flagged tableaux.** Fix λ,ϵ as in the previous section. We will see that $\Omega(\lambda,\epsilon,N)$ naturally enumerates flagged semistandard tableaux. By taking complements $T_{i,j} \mapsto N+1-T_{i,j}$, we have that $\Omega(\lambda,\epsilon,N)$ counts ϵ-*tableaux*, defined as fillings of λ with integers in $\{1,\dots,N+1\}$ weakly increasing in rows and columns, with strict increases forced by e,f. Let $\mathcal{T}(\lambda,\epsilon,N)$ be the set of ϵ-tableaux with entries at most $N+1$; by definition $|\mathcal{T}(\lambda,\epsilon,N)| = \Omega(\lambda,\epsilon,N)$.

Write $\lambda = (p_1^{m_1} > p_2^{m_2} > \cdots > p_r^{m_r})$ as before, and define $M_q = m_1+\cdots+m_q$ for $q = 1\dots,r$. Define the partial sums

$$\begin{cases} E_i = E_i(\epsilon) \coloneqq \sum_{k=1}^{i-1} e_k \text{ for } i = 1,\dots,l,\\ F_j = F_j(\epsilon) \coloneqq \sum_{k=1}^{j-1} f_k \text{ for } j = 1,\dots,\lambda_1.\end{cases}$$

Also consider $\bar{E}_i = i-1-E_i$ and $\bar{F}_j = j-1-F_j$. We remark that $\mathcal{T}(\lambda,\epsilon,N)\neq\emptyset$ if and only if

$$N \geq F_{p_q} + E_{M_q} \text{ for } q = 1,\dots,r. \tag{7.4}$$

Informally put, the quantity $F_{p_q} + E_{M_q}$ counts the number of strict increases that are forced in going from the top left cell of λ to the corner cell in column p_q. For the ϵ-tableau on the left in Figure 3, the E and F vectors are given by $(0,0,1,1,1)$ and $(0,0,1,1,1,2,2)$ respectively, and their barred analogues are given by $(0,1,1,2,3)$ and $(0,1,1,2,3,3,4)$.

We want to transform tableaux in $\mathcal{T}(\lambda,\epsilon,N)$ into semistandard Young tableaux, that is $(1^{l-1},0^{\lambda_1-1})$-tableaux. The general idea is to decrease values in the columns to the right of a strict condition $f_j = 1$, and to increase the values in the rows below a weak condition $e_i = 0$. This leads to the following definition.

Definition 7.6. Fix an ϵ-tableau $T\in\mathcal{T}(\lambda,\epsilon,N)$. We define $\mathrm{Str}(T) = T'$ to be the filling of λ given by

$$T'_{i,j} = T_{i,j} - F_j + \bar{E}_i \quad \text{for all } (i,j)\in\lambda.$$

The ϵ-tableau on the left in Figure 3 belongs to $\mathcal{T}(\lambda,\epsilon,N)$ for $\lambda = (7,7,6,3,3)$, $\epsilon = (0100, 010010)$, and $N = 6$. Its image under Str is depicted on the right using the E and F computed earlier. Proposition 7.7 states that Str is bijective between $\mathcal{T}(\lambda,\epsilon,7)$ and $\mathrm{SSYT}(\lambda;(6^2,6^1,9^2))$.

It is easily checked that $T' = \mathrm{Str}(T)$ is a semistandard Young tableau. Indeed checking that the columns of T' are strictly increasing amounts to showing that $e_i < T_{i+1,j} - T_{i,j} + 1$, whereas showing that the rows are weakly decreasing is equivalent to $f_j \leq T_{i,j+1} - T_{i,j}$. Both these inequalities are immediate. We now work out what the condition that the maximal entry in T is at most $N+1$ becomes under the mapping Str.

Define $\phi_{\epsilon,N} \coloneqq (\phi_1^{m_1},\dots,\phi_r^{m_r})$ by

$$\phi_q = N+1-F_{p_q} + \bar{E}_{M_q} \tag{7.5}$$

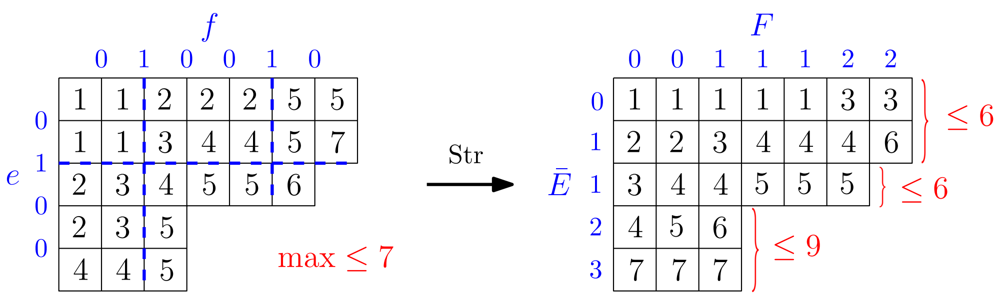


FIGURE 3. The ϵ-tableau coming from the ϵ-partition of Figure 2 (left), and its image under Str (right). The bounds in red indicate constraints of tableaux for which Str is bijective, cf. Proposition 7.7.

for $q = 1\ldots, r$. It follows that for $1 \leq q \leq r-1$,

$$\delta_q := \phi_{q+1} - \phi_q = (\bar{E}_{M_{q+1}} - \bar{E}_{M_q}) + (F_{p_q} - F_{p_{q+1}}) \tag{7.6}$$

is equal to the number of zeros in e between rows M_q and M_{q+1} plus the number of ones in f between columns p_{q+1} and p_q. Therefore $\phi_{\epsilon,N}$ satisfies the inequalities (7.2).

Furthermore, the inequalities (7.4) become $\phi_q \geq 1 + E_{M_q} + \bar{E}_{M_q} = M_q$ for $q \geq 1$, which is precisely the inequalities (7.1). We invite the reader to check that in our running example, we have that $\phi_1 = 7-2+1$, $\phi_2 = 7-2+1$, and $\phi_3 = 7-1+3$. This means that $\phi_{\epsilon,N} = (6^2, 6^1, 9^2)$.

Proposition 7.7. *Given ϵ and N satisfying* (7.4)*, $(\lambda, \phi_{\epsilon,N})$ corresponds to a vexillary permutation w. Furthermore,* Str *is a bijection between $\mathcal{T}(\lambda,\epsilon,N)$ and* $\mathrm{SSYT}(\lambda, \phi_{\epsilon,N})$.

Proof. We have already checked that the inequalities of Proposition 7.2 were satisfied under the hypotheses. It is also clear that Str is well-defined, and that $U_{i,j} \mapsto U_{i,j} + F_j - \bar{E}_i$ provides the desired inverse. □

7.4. **Combinatorial interpretation of a_w.** Let w be a vexillary permutation of shape $\lambda \vdash n-1$ and flag ϕ. From Proposition 7.2, $w = 1^m \times u$ with u indecomposable and vexillary. Clearly $\lambda(u) = \lambda$, while $\phi(w)$ is obtained from $\phi(u)$ by adding m to each entry; let us write this $\phi(w) = m + \phi(u)$ in short. We thus have

$$\nu_u(m) = |\mathrm{SSYT}(\lambda, m + \phi(u))|. \tag{7.7}$$

The next lemma provides some converse to Proposition 7.7.

Lemma 7.8. *Let u be indecomposable and vexillary. There exists a signature ϵ_u on $\lambda(u)$ and a nonnegative integer N_u such that $\phi(u) = \phi_{\epsilon_u,N_u}$. Moreover N_u is given by*

$$N_u = \max_q (F_{p_q}(\epsilon_u) + E_{M_q}(\epsilon_u)).$$

Proof. Let $\phi := \phi(u)$, $\lambda := \lambda(u)$. Also, like before $l = \ell(\lambda)$. We claim that there exist $(e_1,\ldots,e_{l-1}) \in \{0,1\}^{l-1}$ and $(f_1,\ldots,f_{\lambda_1-1}) \in \{0,1\}^{\lambda_1-1}$ such that

$$\sum_{M_q \leq i \leq M_{q+1}-1} (1-e_i) + \sum_{p_{q+1} \leq j \leq p_q - 1} f_j = \phi_{q+1} - \phi_q \tag{7.8}$$

has solutions for all $1 \leq q \leq r-1$. Indeed, as u is vexillary, the inequalities (7.2) state that for any $1 \leq q \leq r-1$, we have $\phi_{q+1} - \phi_q \leq m_{q+1} + p_q - p_{q+1}$. Now, in (7.8), the first sum runs over m_{q+1} elements, whereas the second sum runs over $p_q - p_{q+1}$ elements. It therefore follows that we can pick $e_{M_q}, \ldots, e_{M_{q+1}-1}, f_{p_{q+1}}, \ldots, f_{p_q-1}$ in $\{0,1\}$ such that (7.8) is satisfied. In fact, there are in general many such choices. Having made these choices for $1 \leq q \leq r-1$, we subsequently pick $e_1, \ldots, e_{M_1-1}, f_1, \ldots, f_{p_1-1}$ arbitrary to obtain $(e_1, \ldots, e_{l-1})$ and $(f_1, \ldots, f_{\lambda_1-1})$.

These choices comprise our signature ϵ_u. Indeed, it is readily checked that (7.8) is (7.6) in disguise. Now define $\phi' = \phi_{\epsilon_u, N_u}$ with the value of N_u in the lemma. There is thus an equality in (7.4) for a certain $q \in [r]$, which translates to an equality in (7.1) for the same q. This shows that the vexillary permutation determined by the flag ϕ' does not have 1 as a fixed point. It is therefore equal to u, and it follows that $\phi' = \phi$ as wanted. □

Example 7.9. Consider $u = 346215$ with shape $\lambda = (3^1, 2^2, 1^1)$ and $\phi(u) = (3^1, 3^2, 4^1)$. We then have $(p_1, p_2, p_3) = (3, 2, 1)$ and $(M_1, M_2, M_3) = (1, 3, 4)$. The sequences (e_1, e_2, e_3) and (f_1, f_2) which comprise the signature ϵ_u need to satisfy $(1 - e_3) + f_1 = 1$ and $(1 - e_2) + (1 - e_1) + f_2 = 0$. Thus, we may pick $(e_1, e_2, e_3) = (1, 1, 0)$, and $(f_1, f_2) = (0, 0)$. The corresponding E and F vectors are therefore $(0, 1, 2, 2)$ and $(0, 0, 0)$ respectively. It follows that N_u is $\max\{0+0, 0+2, 0+2\} = 2$.

Theorem 7.10. *Let $u \in \mathrm{S}_{p+1}$ of shape $\lambda \vdash n-1$ be an indecomposable vexillary permutation, and choose ϵ_u, N_u as in Lemma 7.8. Moreover, let $\omega_u := \omega_{\epsilon_u}$ be an ϵ_u-compatible labeling as defined in Section 7.2.*

Let $m \in \{0, \ldots, n-p-1\}$ and consider the permutation $u^{[m]} \in \mathrm{S}'_n$ defined by $u^{[m]} = 1^m \times u \times 1^{n-p-1}$. Then we have

$$\sum_{j \geq 0} \nu_u(j) t^j = \frac{\sum_{T \in \mathrm{SYT}(\lambda)} t^{\mathrm{des}(T; \omega_u) - N_u}}{(1-t)^n}.$$

Proof. We have

$$\nu_u(j) = |\mathrm{SSYT}(\lambda; j + \phi(u))| = |\mathrm{SSYT}(\lambda; j + \phi_{\epsilon_u, N_u})| = |\mathrm{SSYT}(\lambda; \phi_{\epsilon_u, j+N_u})|,$$

and so by Proposition 7.7 we get

$$\nu_u(j) = |\mathcal{T}(\lambda; \epsilon_u, j + N_u)| = \Omega(\lambda; \epsilon_u, j + N_u - 1),$$

and therefore

$$\sum_{j \geq 0} \nu_u(j) t^j = \sum_{j \geq 0} \Omega(\lambda; \epsilon_u, j + N_u) t^j = t^{-N_u} \sum_{j \geq 0} \Omega(\lambda; \epsilon_u, j) t^j,$$

because $\Omega(\lambda; \epsilon_u, j) = 0$ for $j < N_u$. From (7.3) the desired identity follows. □

Comparing the content of Theorem 7.10 with (5.9) from Theorem 5.8 gives the following as an immediate corollary:

Corollary 7.11. *We keep the notations from Theorem 7.10. Then $a_{u^{[m]}}$ is equal to the number of tableaux $T \in \mathrm{SYT}(\lambda)$ with $m + N_u$ ω_{ϵ_u}-descents.*

Example 7.12. We follow up on Example 7.9. The next figure depicts a possible ω_{ϵ_u}.

$$\begin{array}{|c|c|c|} \hline 5 & 6 & 7 \\ \hline 3 & 4 \\ \cline{1-2} 1 & 2 \\ \cline{1-2} 8 \\ \cline{1-1} \end{array}$$

Here are the three standard Young tableaux with exactly two ω_ϵ-descents, coming from the shaded boxes.

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 7 \\ \hline 3 & 4 \\ \cline{1-2} 5 & 6 \\ \cline{1-2} 8 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} 6 & 7 \\ \cline{1-2} 8 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 & 7 \\ \cline{1-2} 8 \\ \cline{1-1}\end{array}$$

It follows that $a_{u^{[0]}} = a_{346215789} = 3$. The reader may further verify that

$$\sum_{j\geq 0} \nu_u(j)t^j = \frac{3 + 24t + 34t^2 + 9}{(1-t)^9}.$$

To further demonstrate that we have a family of combinatorial interpretations depending on the choice of ϵ_u (and ω_u), an alternative legitimate choice for $u = 346215$ is the signature $(1,1,1),(1,0)$, for which N_u equals $\max\{1+0, 1+2, 0+3\} = 3$. Suppose we pick ω_u to read 738 62 51 4 going top to bottom, left to right in the Young diagram of shape λ. Here are the three tableaux $\mathrm{SYT}(\lambda)$ with exactly three ω_u-descents.

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 8 \\ \hline 3 & 4 \\ \cline{1-2} 5 & 6 \\ \cline{1-2} 7 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 6 \\ \hline 3 & 4 \\ \cline{1-2} 5 & 7 \\ \cline{1-2} 8 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2} 6 & 7 \\ \cline{1-2} 8 \\ \cline{1-1}\end{array}$$

Let us revisit the Grassmannian and dominant cases in light of our treatment of the vexillary case. We borrow notation that we have used throughout this section.

(1) If u is indecomposable Grassmannian, then the signature $\phi := \phi(u)$ satisfies $\phi_q - \phi_{q-1} = 0$. It follows that we may pick $(e_1, \dots, e_{l-1}) = (1^{l-1})$ and $(f_1, \dots, f_{\lambda_1 - 1}) = (0^{\lambda_1 - 1})$. If we pick ω_ϵ to correspond to the filling of $\lambda := \lambda(w)$ where we place integers from 1 through $|\lambda|$ from bottom to top and left to right, we see that an ω_ϵ-descent is the same as a traditional descent in SYT, thereby recovering Theorem 6.16.

(2) Next consider u dominant. One can see that $(e_1, \dots, e_{l-1}) = (0^{l-1})$ and $(f_1, \dots, f_{\lambda_1 - 1}) = (0^{\lambda_1 - 1})$ give a valid signature. We pick the natural labeling where we place integers from 1 through $|\lambda|$ from top to bottom and left to right, so that an ω_ϵ-descent is a traditional ascent of an SYT.

We remark that *shifted dominant* permutations of the type $1 \times u$ for u dominant occur in a number of articles [7, 22, 63].

Finally, let us briefly sketch why the invariance properties of Proposition 5.1 are apparent in this combinatorial interpretation. Fix $\lambda \vdash n-1$, and let $H_q := m_{q+1} + p_q - p_{q+1}$ for $q = 1, \dots, r-1$ using previously introduced notation. Let $u \in \mathrm{S}_{p+1}$ be an indecomposable vexillary with shape λ and flag differences $\delta_q := \phi_{q+1} - \phi_q$ for $q = 1, \dots, r-1$. Define $\bar{u} = w_o^{p+1} u w_o^{p+1}$ where w_0^{p+1} denotes the longest word in S_{p+1}. Then it follows from [43, Formulas (1.41) and (1.42)] that the indecomposable vexillary permutations $\bar{u}$ and u^{-1} are characterized as follows:

- $\bar{u}$ has shape λ' and flag differences $(\delta_{r-q})_{q=1,\dots,r-1}$;
- u^{-1} has shape λ' and flag differences $(H_{r-q} - \delta_{r-q})_{q=1,\dots,r-1}$.

We fix a signature $\epsilon_u = (e, f)$ and a labeling ω_u for u as in Theorem 7.10. Then the following claims are easily checked:

- A valid signature for $\bar{u}$ is given by $\epsilon_{\bar{u}} := (f, e)$ on λ'. A compatible $\omega_{\bar{u}}$ is defined by $\omega_{\bar{u}}(i,j) := \omega_u(j,i)$ for any $(i,j) \in \lambda'$.
- A valid signature for u^{-1} is given by $\epsilon_{\bar{u}} := (1-f, 1-e)$ on λ' where naturally $(1-f)_j = 1-f_j$ and $(1-e)_i = 1-e_i$. A compatible $\omega_{u^{-1}}$ is defined by $\omega_{u^{-1}} = n - \omega_{\bar{u}}$.

We leave it to the interested reader to show the invariance properties of Proposition 5.1 from the combinatorial interpretation afforded by Corollary 7.11 (the invariance under conjugation by w_o is more involved).

8. Further remarks

8.1. The original motivation for this paper was to investigate a combinatorial interpretation for the numbers a_w. We know from geometry that the numbers a_w are nonnegative, can we find a family of objects counted by a_w? This was achieved in this work for Łukasiewicz permutations (Theorem 6.5) and vexillary permutations (Theorem 7.10).

The hope is to find a combinatorial interpretation in general, from which the various properties established in 5 would be apparent. Note that Theorem 5.6 strongly suggests that a_w counts a subset of the reduced words of w, which in turn hints that the Edelman-Greene correspondence [21] may play a role.

Based on Theorem 5.6, it would be interesting to generalize the results in Section 7 to encompass the whole class of quasiindecomposable permutations.

A natural special case, which generalizes the Grassmannian case, is when w is quasiindecomposable and fully commutative. Since the number of reduced words $\mathbf{i}$ for such a w is the number of SYTs $f^{\lambda/\mu}$ for an appropriate connected skew shape λ/μ with $n-1$ boxes, and all such $\mathbf{i}$ give the same $c(\mathbf{i})$, the question of giving a combinatorial interpretation for a_w amounts to giving one for $\frac{f^{\lambda/\mu}}{(n-1)!}A_{c(\mathbf{i})}$. Also the Schubert polynomial in this case is a flagged skew Schur function, so that ν_w can be interpreted as counting certain flagged skew tableaux; an approach in the manner of Section 7 may be successful. As a curious aside, we remark here that one can derive the hook-content formula for λ/μ by piecing together our Theorem 5.8, Theorem 5.2, and Proposition 4.5.

8.2. Theorems 5.6 and 5.8 give pleasant summation formulas for the numbers a_w. It would be interesting to find a common generalization of them. We note that Theorem 5.8 fails in general: in fact, our data seems to show that as soon as u is not indecomposable, the numerator on the right hand side has at least one negative coefficient.

Another avenue worth exploring, and more in line with the theme of [8] and motivated by Brenti's Poset Conjecture [14], is investigating aspects like real-rootedness, unimodality and log-concavity for the numerators of the right hand side in Theorem 5.8. By work of Brenti [14] and Brändén [12, 13], the Grassmannian case is already well understood.

8.3. Given $w \in \mathrm{S}_\infty$, consider the polynomial $\widetilde{\mathrm{M}}_w(x_1, x_2, \ldots)$ defined by

$$\widetilde{\mathrm{M}}_w := \frac{1}{\ell(w)!}\mathrm{M}_w(x_1, x_1+x_2, x_1+x_2+x_3, \ldots) = \frac{1}{\ell(w)!}\sum_{\mathbf{i}\in \mathrm{Red}(w)} y^{c(\mathbf{i})}.$$

Now let $w \in \mathrm{S}'_n$. It is quite striking to compare the formulas given by the two approaches of Section 3. Indeed by Macdonald's identity (2.15), we have $\widetilde{\mathrm{M}}_w(1,1,\ldots) = \mathfrak{S}_w(1,1,\ldots) = \nu_w$. Also, by Theorems 3.2 and 5.2, we moreover have $\langle \widetilde{\mathrm{M}}_w \rangle_n = \langle \mathfrak{S}_w \rangle_n = a_w$. The coincidence between these specializations certainly deserves an algebraic explanation.

8.4. The summatory results for connected mixed Eulerian numbers (Proposition 4.5) and quasi-indecomposable permutations (Theorem 5.8) can be expressed compactly in terms of certain *back stable* analogues, inspired by the work of Lam, Lee and Shimozono [38].

Consider the algebra $\mathcal{B}$ of bounded degree power series in $\mathbb{Q}[[x_i, i \in \mathbb{Z}]]$ that are polynomials in the $x_i, i > 0$, and symmetric in the $x_i, i \leq 0$. Thus $\mathcal{B}$ identifies naturally with $\Lambda(x_i, i \leq 0) \otimes \mathbb{Q}[x_i, i > 0]$. Let $f \in \mathcal{B}$ be homogeneous of degree $n-1$, written $f \in \mathcal{B}^{(n-1)}$. Following [38], consider the truncation operator $\pi_+(f) := f(\dots, 0, x_1, x_2, \dots)$ and the shift operator γ that sends $x_i \mapsto x_{i+1}$ for all all $i \in \mathbb{Z}$. This given, define $f[m] := \pi_+(\gamma^m(f))$ which is a polynomial in $x_1, x_2, \dots$, and let $f[m](\mathbf{1})$ denote its evaluation when all $x_i, i > 0$ are specialized to 1. Then $f[m](\mathbf{1})$ is a polynomial in m of degree $\leq n-1$ (easy), and we infer the existence of $h_m^f \in \mathbb{Q}$ such that

$$\sum_{j \geq 0} f[j](\mathbf{1}) t^j = \frac{\sum_{m \geq 0} h_m^f t^m}{(1-t)^n}. \tag{8.1}$$

Definition 8.1. Let $\mathcal{D}^n$ be the subspace of $f \in \mathcal{B}^{(n-1)}$ such that $h_m^f = \langle f[m] \rangle_n$ for any $m \geq 0$.

We now briefly touch upon some elements that lie $\mathcal{D}^n$ by our results. First, Theorem 5.8 says that the *back stable Schubert polynomial* $\overleftarrow{\mathfrak{S}_u}$ [38] is in $\mathcal{D}^n$ if u is indecomposable of length $n-1$.

Additionally, if f is a symmetric function in the $x_i, i < 0$, then $f[m]$ is the symmetric polynomial $f(x_1, \dots, x_m)$. The fact that $f \in \mathcal{D}^n$ is one of the main results of [48].

Let $\overleftarrow{y_k}$ be the series $\overleftarrow{y_k} = \dots + x_{-2} + x_{-1} + x_0 + \dots + x_{k-1} + x_k = \sum_{i \leq k} x_i$. Given $\mathbf{a} \in \mathcal{W}_p^{(n-1)}$, define $\overleftarrow{y_{\mathbf{a}}} = \overleftarrow{y_1}^{a_1} \overleftarrow{y_2}^{a_2} \cdots \overleftarrow{y_p}^{a_p}$. Then Proposition 4.5 says precisely that if $\mathbf{a}$ is a strong composition, that is $\mathbf{a} \vDash n-1$, then $\overleftarrow{y_{\mathbf{a}}} \in \mathcal{D}^n$.

In view of the aforementioned, the following problem is natural: *Characterize the space $\mathcal{D}^n$, for instance by finding a distinguished basis.*

8.5. By expanding a double Schubert polynomial in terms of Schubert polynomials (cf. [44]), Formula (3.3) gives

$$\Sigma_h = \sum_{\substack{u,v \in S_n \\ v^{-1}u = w_h \\ \ell(u)+\ell(v)=\ell(w_h)}} \mathfrak{S}_u \mathfrak{S}_v \mod I_n. \tag{8.2}$$

In [5], this latter formula is used to give an explicit expansion of Σ_h in the Schubert basis in the easy special case where $w_h \in \mathrm{S}_k \subset \mathrm{S}_n$ with $2k \leq n$.

In the case $h = (2, 3, \dots, n, n)$ that is the subject of our study, we have $w_h = w_o^{n-1}$, so we get

$$\tau_n = \sum_{\substack{u,v \in \mathrm{S}_n \\ uv^{-1} = w_o^{n-1} \\ \ell(u)+\ell(v) = \binom{n-1}{2}}} \sigma_u \sigma_{w_o v w_o}$$

We may simplify the summation range: as shown in [27, Lemma 6.1], the conditions are equivalent to $u \in \mathrm{S}_{n-1}$ (and $v = w_o^{n-1} u$). Let us give a short proof: For any $u \in \mathrm{S}_n$, $\ell(u) + \ell(w_o^{n-1} u) \geq \ell(w_o^{n-1}) = \binom{n-1}{2}$, since any pair (i, j) with $1 \leq i < j \leq n-1$ is an inversion in either u or $w_o^{n-1} u$.

It follows then that $\ell(u)+\ell(w_o^{n-1}u)=\ell(w_o^{n-1})$ if no pair (i,n) is an inversion either u or $w_o^{n-1}u$, which is clearly equivalent to $u(n)=n$ so that $u\in \mathrm{S}_{n-1}$. Therefore we can write

$$\tau_n=\sum_{u\in \mathrm{S}_{n-1}}\sigma_u\sigma_{1\times w_o^{n-1}u}.$$

Extracting coefficients gives the summation formulas for $w\in \mathrm{S}'_n$:

$$a_w=\sum_{u\in \mathrm{S}_{n-1}}c^w_{u,1\times w_o^{n-1}u}, \tag{8.3}$$

where the structure coefficients $c^w_{u,v}$ are defined in (2.8). Together with the combinatorial interpretations (Theorem 6.5, Corollary 7.11) and our various other results about the a_w, Equation (8.3) gives information about certain coefficients c^w_{uv} that may be of interest in the quest to find a combinatorial interpretation for them.

8.6. To go beyond the focus of this work, a natural endeavour is to compute the coefficients in the Schubert basis for the other regular Hessenberg classes Σ_h, see Section 3.2.

As mentioned above, this was essentially done in [5] for the case $w_h\in \mathrm{S}_k\subset \mathrm{S}_n$ with $2k\leq n$; they also consider the case where $h(i)=n$ for $i>1$. The starting point is the formula (3.3) for Σ_h.

Let us also mention the work [34] which gives another polynomial representative for Σ_h: consider the permutation $w'_h\in \mathrm{S}_{2n}$ given by $w'_h(i+h(i))=n+i$ for $i\in[n]$ and put the values $1,\ldots,n$ from left to right in the remaining entries. Then

$$\Sigma_h=\mathfrak{S}_{w'_h}(x_1,\ldots,x_{h(1)},x_1,x_{h(1)+1},\ldots,x_{h(2)},x_2,x_{h(2)+1},\ldots,x_{h(n)},x_n) \mod I_n \tag{8.4}$$

We would also like to emphasize the recent work of Kim [36]: he investigates a larger family of cohomology classes, in all types, coming from varieties related to the Deligne-Lusztig varieties. His formulas in type A extend those of [5].

8.7. Although this work is focused on the combinatorics of type A, the setting makes sense in all types, and certain of our results can be generalized. The starting point is again Klyachko's work [37].

Fix G a complex reductive group, B a Borel subgroup and T a maximal torus inside B. Let Φ be the corresponding root system of rank n, with Weyl group $W:=N_G(T)/T$. Let $X(\Phi)$ be the n-dimensional toric variety determined by the Coxeter arrangement determined by Φ, and let a_w^Φ the coefficients of $[X(\Phi)]$ in the cohomology of the generalized flag variety G/B:

$$[X(\Phi)]=\sum_{w\in W'}a_w^\Phi\sigma_{w_ow},$$

where $W'\subset W$ consist of the elements of length n. Then one can deduce expressions analogous to Theorems 3.1 and 5.2, using Klyachko's results and the mixed Eulerian numbers $A^\Phi_{c(\mathbf{i})}$ introduced by Postnikov [50]. Positivity of a_w^Φ and stability under inverse follow again easily from such formulas for instance. This is work in progress.

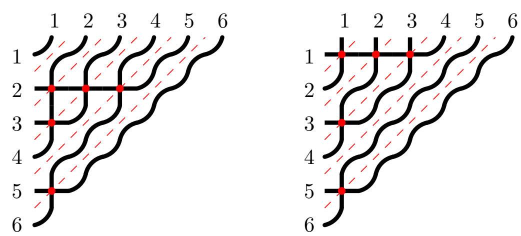
Appendix A. Proof of Proposition 6.4

Let $w \in \mathrm{S}_n$ with code $(c_1, \ldots, c_{n-1})$. We define the composition $\bar{a}(w) = (a_1, \ldots, a_n)$ by

$$a_i = |\{1 \leq j \leq i \mid c_j > i - j\}|. \tag{A.1}$$

More generally, consider $\gamma \in \mathrm{PD}(w)$. Following [62], let $a(\gamma) = (a_1, a_2, \ldots, a_n)$ where a_i is the number of $+$'s on the kth antidiagonal $i + j = k - 1$. Then $\bar{a}(w) = a(\gamma_w)$ where w is the bottom pipe dream of w.

Example A.1. For $w = 153264$ we have $\mathrm{code}(w) = (0, 3, 1, 0, 1, 0)$ and $\bar{a}(w) = (0, 1, 2, 1, 1, 0)$, while if $w = 413265$, then $\mathrm{code}(w) = (3, 0, 1, 0, 1, 0)$ and $\bar{a}(w) = (1, 1, 2, 0, 1, 0)$. For the first permutation, neither $\mathrm{code}(w) = (0, 3, 1, 0, 1, 0)$ nor $\bar{a}(w)$ are in $\mathcal{LC}_n$, while both of them are in $\mathcal{LC}_n$ in the second case. Refer to the diagram that follows.



Proposition A.2. *For $w \in \mathrm{S}'_n$, we have that* $\mathrm{code}(w) \in \mathcal{LC}_n$ *if and only if* $\bar{a}(w) \in \mathcal{LC}_n$.

Proof. Write $\mathrm{code}(w) = (c_1, \ldots, c_n)$ and $\bar{a}(w) = (a_1, \ldots, a_n)$. For $1 \leq i \leq n - 1$, we have

$$\sum_{1 \leq j \leq i} a_j = \sum_{1 \leq j \leq i} \min\{c_j, i - j + 1\} \leq \sum_{1 \leq j \leq i} c_j. \tag{A.2}$$

It follows immediately that if $\bar{a}(w) \in \mathcal{LC}_n$ then $c(w) \in \mathcal{LC}_n$.

Conversely, assume $\bar{a}(w) \notin \mathcal{LC}_n$, so that there exists $1 \leq k \leq n - 1$ such that

$$\sum_{1 \leq j \leq k} a_j < k. \tag{A.3}$$

Let k be the smallest integer with this property. This forces $\sum_{1 \leq j \leq k-1} a_j = k - 1$ and $a_k = 0$ (note that this holds in the special case $k = 1$ also). By (A.1) this implies in turn that $c_j \leq k - j$ for $j = 1, \ldots, k$ and thus, by using the leftmost equality in (A.2),

$$\sum_{1 \leq j \leq k} c_j = \sum_{1 \leq j \leq k} a_j = k - 1. \tag{A.4}$$

Therefore $\mathrm{code}(w) \notin \mathcal{LC}_n$, which finishes the proof. $\square$

Proof of Proposition 6.4. We use here [62, Lemma 3.6(iii)] which states that for any $w \in \mathrm{S}_\infty$, $a(\gamma_w) = a(\gamma_{w^{-1}})$, which translates into $\bar{a}(w) = \bar{a}(w^{-1})$. We then conclude by Proposition A.2. $\square$

Acknowledgments

We are grateful to Sara Billey, Sean Griffin, and particularly Alex Woo for numerous helpful discussions at the early stages of this project. We would also like to thank Marcelo Aguiar, Dave Anderson, Jang Soo Kim, Alejandro Morales, Allen Knutson, and Alex Yong for enlightening discussions.

References

[1] H. Abe, L. DeDieu, F. Galetto, and M. Harada. Geometry of Hessenberg varieties with applications to Newton-Okounkov bodies. *Selecta Math. (N.S.)*, 24(3):2129–2163, 2018. 2, 11

[2] H. Abe and T. Horiguchi. A survey of recent developments on hessenberg varieties, 2019, arXiv:1904.11155. 1

[3] H. Abe, T. Horiguchi, and M. Masuda. The cohomology rings of regular semisimple Hessenberg varieties for $h = (h(1), n, \ldots, n)$. *J. Comb.*, 10(1):27–59, 2019. 1

[4] T. Abe, T. Horiguchi, M. Masuda, S. Murai, and T. Sato. Hessenberg varieties and hyperplane arrangements, 2016, arXiv:1611.00269. 1

[5] D. Anderson and J. Tymoczko. Schubert polynomials and classes of Hessenberg varieties. *J. Algebra*, 323(10):2605–2623, 2010. 1, 2, 3, 10, 11, 15, 31, 32

[6] N. Bergeron and S. Billey. RC-graphs and Schubert polynomials. *Experiment. Math.*, 2(4):257–269, 1993. 9

[7] N. Bergeron, C. Ceballos, and V. Pilaud. Hopf dreams and diagonal harmonics, 2018, arXiv:1807.03044. 29

[8] A. Berget, H. Spink, and D. Tseng. Log-concavity of matroid h-vectors and mixed Eulerian numbers, 2020, arXiv:2005.01937. 14, 15, 30

[9] S. C. Billey, A. E. Holroyd, and B. J. Young. A bijective proof of Macdonald's reduced word formula. *Algebr. Comb.*, 2(2):217–248, 2019. 9

[10] S. C. Billey, W. Jockusch, and R. P. Stanley. Some combinatorial properties of Schubert polynomials. *J. Algebraic Combin.*, 2(4):345–374, 1993. 1, 5, 8, 9

[11] A. Borel. Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts. *Ann. of Math. (2)*, 57:115–207, 1953. 1, 7

[12] P. Brändén. On operators on polynomials preserving real-rootedness and the Neggers-Stanley conjecture. *J. Algebraic Combin.*, 20(2):119–130, 2004. 30

[13] P. Brändén. Sign-graded posets, unimodality of W-polynomials and the Charney-Davis conjecture. *Electron. J. Combin.*, 11(2):Research Paper 9, 15, 2004/06. 30

[14] F. Brenti. Unimodal, log-concave and Pólya frequency sequences in combinatorics. *Mem. Amer. Math. Soc.*, 81(413):viii+106, 1989. 30

[15] M. Brion. Lectures on the geometry of flag varieties. In *Topics in cohomological studies of algebraic varieties*, Trends Math., pages 33–85. Birkhäuser, Basel, 2005. 6

[16] P. Brosnan and T. Y. Chow. Unit interval orders and the dot action on the cohomology of regular semisimple Hessenberg varieties. *Adv. Math.*, 329:955–1001, 2018. 1

[17] A. Bălibanu. The Peterson variety and the wonderful compactification. *Represent. Theory*, 21:132–150, 2017. 2

[18] F. De Mari, C. Procesi, and M. A. Shayman. Hessenberg varieties. *Trans. Amer. Math. Soc.*, 332(2):529–534, 1992. 1

[19] F. De Mari and M. A. Shayman. Generalized Eulerian numbers and the topology of the Hessenberg variety of a matrix. *Acta Appl. Math.*, 12(3):213–235, 1988. 2

[20] E. Drellich. Monk's rule and Giambelli's formula for Peterson varieties of all Lie types. *J. Algebraic Combin.*, 41(2):539–575, 2015. 2

[21] P. Edelman and C. Greene. Balanced tableaux. *Adv. in Math.*, 63(1):42–99, 1987. 30

[22] L. Escobar and K. Mészáros. Toric matrix Schubert varieties and their polytopes. *Proc. Amer. Math. Soc.*, 144(12):5081–5096, 2016. 29

[23] S. Fomin and R. P. Stanley. Schubert polynomials and the nil-Coxeter algebra. *Adv. Math.*, 103(2):196–207, 1994. 1

[24] W. Fulton. *Young tableaux*, volume 35 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1997. With applications to representation theory and geometry. 6

[25] S. Griffin. Schubert-Peterson intersection via Gröbner geometry, 2019, Personal communication. 21
[26] Z. Hamaker, O. Pechenik, D. E Speyer, and A. Weigandt. Derivatives of Schubert polynomials and proof of a determinant conjecture of Stanley. *Algebraic Combinatorics*, 3(2):301–307, 2020. 9
[27] M. Harada, T. Horiguchi, M. Masuda, and S. Park. The volume polynomial of regular semisimple Hessenberg varieties and the Gelfand-Zetlin polytope. *Proc. Steklov Inst. Math.*, 305:318–344, 2019. 1, 2, 8, 15, 31
[28] M. Harada and M. Precup. The cohomology of abelian Hessenberg varieties and the Stanley-Stembridge conjecture. *Sém. Lothar. Combin.*, 80B:Art. 49, 12, 2018. 1
[29] M. Harada and J. Tymoczko. A positive Monk formula in the S^1-equivariant cohomology of type A Peterson varieties. *Proc. Lond. Math. Soc. (3)*, 103(1):40–72, 2011. 2
[30] M. Harada and J. Tymoczko. Poset pinball, GKM-compatible subspaces, and Hessenberg varieties. *J. Math. Soc. Japan*, 69(3):945–994, 2017. 1
[31] J. Huh and E. Katz. Log-concavity of characteristic polynomials and the Bergman fan of matroids. *Math. Ann.*, 354(3):1103–1116, 2012. 2
[32] E. Insko. Schubert calculus and the homology of the Peterson variety. *Electron. J. Combin.*, 22(2):Paper 2.26, 12, 2015. 2
[33] E. Insko and J. Tymoczko. Intersection theory of the Peterson variety and certain singularities of Schubert varieties. *Geom. Dedicata*, 180:95–116, 2016. 2
[34] E. Insko, J. Tymoczko, and A. Woo. A formula for the cohomology and K-class of a regular Hessenberg variety, 2020. 32
[35] E. Insko and A. Yong. Patch ideals and Peterson varieties. *Transform. Groups*, 17(4):1011–1036, 2012. 2
[36] D. Kim. Homology Class of a Deligne–Lusztig Variety and Its Analogs. *Int. Math. Res. Not. IMRN*, (4):1246–1280, 2020. 32
[37] A. A. Klyachko. Orbits of a maximal torus on a flag space. *Funktsional. Anal. i Prilozhen.*, 19(1):77–78, 1985. 2, 10, 22, 32
[38] T. Lam, S.J. Lee, and M. Shimozono. Back stable Schubert calculus, 2018, arXiv:1806.11233. 31
[39] A. Lascoux and M.-P. Schützenberger. Polynômes de Schubert. *C. R. Acad. Sci. Paris Sér. I Math.*, 294(13):447–450, 1982. 1, 7, 23, 24
[40] G. Liu. Mixed volumes of hypersimplices. *Electron. J. Combin.*, 23(3):Paper 3.19, 19, 2016. 13
[41] M. Lothaire. *Combinatorics on words*, volume 17 of *Encyclopedia of Mathematics and its Applications*. Addison-Wesley Publishing Co., Reading, Mass., 1983. 19
[42] M. Lothaire. *Applied combinatorics on words*, volume 105 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2005. 20
[43] I. G. Macdonald. *Notes on Schubert polynomials*. Publications du LaCIM, 1991. 9, 24, 25, 29
[44] L. Manivel. *Symmetric functions, Schubert polynomials and degeneracy loci*, volume 6 of *SMF/AMS Texts and Monographs*. American Mathematical Society, Providence, RI; Société Mathématique de France, Paris, 2001. Translated from the 1998 French original by John R. Swallow, Cours Spécialisés [Specialized Courses], 3. 6, 7, 9, 11, 20, 22, 23, 31
[45] B. H. Margolius. Permutations with inversions. *J. Integer Seq.*, 4(2):Article 01.2.4, 13, 2001. 20
[46] D. Monk. The geometry of flag manifolds. *Proc. London Math. Soc. (3)*, 9:253–286, 1959. 16
[47] A. H. Morales, I. Pak, and G. Panova. Hook formulas for skew shapes III. Multivariate and product formulas. *Algebr. Comb.*, 2(5):815–861, 2019. 9
[48] P. Nadeau and V. Tewari. Divided symmetrization and quasisymmetric functions, 2019, arXiv:1908.10934. 3, 13, 19, 22, 31
[49] F. Petrov. Combinatorial and probabilistic formulae for divided symmetrization. *Discrete Math.*, 341(2):336–340, 2018. 13, 14, 19
[50] A. Postnikov. Permutohedra, associahedra, and beyond. *Int. Math. Res. Not. IMRN*, (6):1026–1106, 2009. 2, 3, 12, 13, 14, 15, 19, 32
[51] A. Postnikov and R. P. Stanley. Chains in the Bruhat order. *J. Algebraic Combin.*, 29(2):133–174, 2009. 8
[52] C. Procesi. The toric variety associated to Weyl chambers. In *Mots*, Lang. Raison. Calc., pages 153–161. Hermès, Paris, 1990. 2
[53] K. Rietsch. Totally positive Toeplitz matrices and quantum cohomology of partial flag varieties. *J. Amer. Math. Soc.*, 16(2):363–392, 2003. 2

[54] J. Shareshian and M. L. Wachs. Chromatic quasisymmetric functions and Hessenberg varieties. In *Configuration spaces*, volume 14 of *CRM Series*, pages 433–460. Ed. Norm., Pisa, 2012. 1

[55] J. Shareshian and M. L. Wachs. Chromatic quasisymmetric functions. *Adv. Math.*, 295:497–551, 2016. 1

[56] N. J. A. Sloane. The encyclopedia of integer sequences. 5, 16

[57] R. P. Stanley. On the number of reduced decompositions of elements of Coxeter groups. *European J. Combin.*, 5(4):359–372, 1984. 23

[58] R. P. Stanley. *Enumerative combinatorics. Vol. 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997. With a foreword by Gian-Carlo Rota, Corrected reprint of the 1986 original. 17, 18, 25

[59] R. P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin. 25

[60] J. S. Tymoczko. Paving Hessenberg varieties by affines. *Selecta Math. (N.S.)*, 13(2):353–367, 2007. 1

[61] M. L. Wachs. Flagged Schur functions, Schubert polynomials, and symmetrizing operators. *J. Combin. Theory Ser. A*, 40(2):276–289, 1985. 24

[62] A. E. Weigandt. Schubert polynomials, 132-patterns, and Stanley's conjecture. *Algebraic Combinatorics*, 1(4):415–423, 2018. 9, 33

[63] A. Woo. Catalan numbers and schubert polynomials for $w = 1(n+1)...2$, 2004, arXiv:math/0407160. 29

Univ Lyon, CNRS, Université Claude Bernard Lyon 1, UMR 5208, Institut Camille Jordan, 43 blvd. du 11 novembre 1918, F-69622 Villeurbanne cedex, France
Email address: `nadeau@math.univ-lyon1.fr`

Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104, USA
Email address: `vvtewari@math.upenn.edu`