MATH 644, FALL 2011, HOMEWORK 3

Exercise 1. *(Weyl's lemma) [10 points, or 15 points (5 points extra credit)] In this exercise, we outline the proof of Weyl's lemma, which is a generalization of the Theorem we proved in class that states that all (C^2) harmonic functions are smooth, or more generally that all continuous functions which satisfy the mean value property are smooth.*

The claim which we prove is:

Lemma 0.1. *Let $U \subseteq \mathbb{R}^n$ be open and bounded. Suppose that $u : U \to \mathbb{R}$ and $u \in L^1_{loc}(U)$. Furthermore, suppose that:*

$$\int_U u(x)\Delta\phi(x)dx = 0$$

for all $\phi \in C_0^\infty(U)$. Then u is harmonic on U, and in particular it is smooth.

We note that harmonic functions on U indeed satisfy the condition from the Lemma by integration by parts.

If one shows this fact for $u \in C(U)$, this counts for the full 10 points (on an earlier version of the assignment). If one shows the claim for $u \in L^1_{loc}(U)$, this counts for an additional 5 points of extra credit.

a) Given $\epsilon > 0$, let us consider $\phi \in C_0^\infty(U)$ which are supported inside $U_\epsilon := \{x \in U; dist(x, \partial U) > \epsilon\}$. Let $u^\epsilon := u * \eta_\epsilon$, where $\eta_\epsilon := \frac{1}{\epsilon^n}\eta(\frac{\cdot}{\epsilon})$ is the mollifier constructed in class. Show that u^ϵ is harmonic on U_ϵ.

b) Show that, for all $\epsilon > 0$, one has:

$$\int_{U_\epsilon} |u^\epsilon(x)|dx \leq \int_U |u(x)|dx.$$

c) We fix an $R > 0$ and we consider $V := \overline{U_R}$. Show that, for $0 < \epsilon < \frac{R}{2}$, $|u^\epsilon|$ is bounded on V with a bound independent of ϵ. (HINT: Recall the Mean Value Property).

d) Show that u_ϵ is equicontinuous on V.

e) Use the Arzela-Ascoli Theorem and show that we can find $v \in C(\overline{V})$ such that, up to a subsequence, u_ϵ converges to v uniformly on $\overline{V}$, as $\epsilon \to 0$. Why is v harmonic on V?

f) Recall that $u_\epsilon \to u$ on U (you are allowed to use the L^1_{loc} version of this statement, which was stated in class). Deduce that u is harmonic on U.

Exercise 2. *(An estimate for solutions to Poisson's equation) [Evans, Problem 6 from Chapter 2; 5 points] Suppose that U is a bounded, open subset of $\mathbb{R}^n$ and that $u \in C^2(U) \cap C(\bar{U})$ solves:*

$$\begin{cases} -\Delta u = f, \text{ on } U \\ u = g, \text{ on } \partial U \end{cases} \tag{1}$$

for $f \in C(\bar{U}), g \in C(\partial U)$. Show that there exists a constant $C > 0$ which depends only on U such that:

$$\max_{\bar{U}} |u| \leq \max_{\partial U} |g| + C \max_{\bar{U}} |f|.$$

(HINT: Look at the function $u + \lambda|x|^2$ for an appropriate value of λ and use properties of subharmonic functions from last week's homework.)

Exercise 3. *(An example of non-smoothness at the boundary of a harmonic function)[Evans, Problem 9 from Chapter 2; 5 points] Let u be the solution of*

$$\begin{cases} \Delta u = 0, \ on\, \mathbb{R}^n_+ \\ u = g, \ on\, \partial\mathbb{R}^n_+ \end{cases} \tag{2}$$

given by Poisson's formula for half-space. Suppose that $g \in C(\partial\mathbb{R}^n_+) \cap L^\infty(\partial\mathbb{R}^n_+)$ is non-negative and that $g(x) = |x|$ for $|x| \leq 1$. Calculate the limit: $\lim_{\lambda\to 0} \frac{u(\lambda e_n)-u(0)}{\lambda}$. Deduce that u is not smooth up to $\partial\mathbb{R}^n_+$.

This homework assignment is due in class on Wednesday, October 5. The first problem is worth 10 points, whereas the second two problems are worth 5 points. Good Luck!