MATH 644, FALL 2011, HOMEWORK 2

Exercise 1. *(A mean value formula for the Poisson equation on $B(0,r)$) [Evans, Problem 3 in Chapter 2] Suppose that $r > 0$, and that $n \geq 3$. Consider $B(0,r) \subseteq \mathbb{R}^n$ and*

$$u \in C^2(B(0,r)) \cap C(\overline{B(0,r)})$$

such that:

$$\begin{cases} -\Delta u = f, \text{ on } B(0,r) \\ u = g, \text{ on } \partial B(0,r). \end{cases} \tag{1}$$

Modify the proof of the mean value formula for harmonic functions to show that:

$$u(0) = \fint_{\partial B(0,r)} g(y)dS(y) + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}}\right) f(x)dx$$

Exercise 2. *(A direct proof of the maximum principle) [Evans, Problem 4 in Chapter 2] Suppose that $U \subseteq \mathbb{R}^n$ is open and bounded and suppose that $u \in C^2(U) \cap C(\overline{U})$ is harmonic on U. By considering the functions $u_\epsilon := u + \epsilon|x|^2$, for $\epsilon > 0$, show that:*

$$\max_{\overline{U}} u = \max_{\partial U} u.$$

Exercise 3. *(Subharmonic functions) [Evans, Problem 5 in Chapter 2] We say that $v \in C^2(\overline{U})$ is **subharmonic** if:*

$$-\Delta v \leq 0.$$

(1) *Prove that for v subharmonic, one has:*

$$v(x) \leq \fint_{B(x,r)} v(y)dy$$

for all $B(x,r) \subseteq U$.

(2) *Show that $\max_{\overline{U}} v = \max_{\partial U} v$.*

(3) *Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume that u is harmonic and take $v := \phi(u)$. Prove that u is subharmonic.*

(4) *Prove that $v := |\nabla u|^2$ is subharmonic whenever u is harmonic.*

Exercise 4. *(A specific class of harmonic functions) Find all polynomials $P(x,y) = \sum_{k=0}^{n} c_k x^k y^{n-k}$ in x, y which are homogeneous of degree n and which are harmonic. Here, $c_k \in \mathbb{C}$.*

Each problem is worth 5 points. This assignment is due in class on Wednesday, September 28. Good Luck!