MATH 425, PRACTICE MIDTERM EXAM 2, SOLUTIONS.

Exercise 1. *Suppose that u solves the boundary value problem:*

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = 1, \text{ for } 0 < x < 1, t > 0 \\ u(x,0) = 0, \text{ for } 0 \leq x \leq 1 \\ u(0,t) = u(1,t) = 0, \text{ for } t > 0. \end{cases} \tag{1}$$

a) Find a function $v = v(x)$ which solves:

$$\begin{cases} -v_{xx}(x) = 1, \text{ for } 0 < x < 1 \\ v(0) = v(1) = 0. \end{cases}$$

b) Show that:

$$u(x,t) \leq v(x)$$

for all $x \in [0,1], t > 0$.

c) Show that:

$$u(x,t) \geq (1 - e^{-2t})v(x)$$

for all $x \in [0,1], t > 0$.

d) Deduce that, for all $x \in [0,1]$:

$$u(x,t) \to v(x)$$

as $t \to \infty$.

Solution:

a) We need to solve $v''(x) = -1$ with boundary conditions $v(0) = v(1) = 0$. The ODE implies that $v(x) = -\frac{1}{2}x^2 + Ax + B$ for some constants A, B. We get the system of linear equations:

$$\begin{cases} B = 0 \\ -\frac{1}{2} + A + B = 0 \end{cases}$$

from where it follows that:

$$A = \frac{1}{2} \text{ and } B = 0.$$

Hence:

$$v(x) = \frac{1}{2}x \cdot (1 - x).$$

b) Let us now think of v as a function of v as a function of (x,t) which doesn't depend on x. By construction, we know that:

$$\begin{cases} v_t(x,t) - v_{xx}(x,t) = 1, \text{ for } 0 < x < 1, t > 0 \\ v(x,0) \geq 0, \text{ for } 0 \leq x \leq 1 \\ v(0,t) = v(1,t) = 0, \text{ for } t > 0. \end{cases}$$

Here, we used the fact that $\frac{1}{2}x \cdot (1 - x) \geq 0$ for $0 \leq x \leq 1$. By using the *Comparison principle* for the heat equation (Exercise 3 on Homework Assignment 4), it follows that:

$$u(x,t) \leq v(x,t) = v(x)$$

for all $x \in [0,1], t > 0$.

c) Let us define:

$$w(x,t) := (1 - e^{-2t})v(x) = \frac{1}{2} \cdot (1 - e^{-2t}) \cdot x(1-x)$$

We compute:

$$w_t(x,t) = e^{-2t} \cdot x(1-x)$$
$$w_{xx}(x,t) = -(1 - e^{-2t}) = -1 + e^{-2t}.$$

Hence:

$$w_t(x,t) - w_{xx}(x,t) = 1 - e^{-2t}\Big(1 - x(1-x)\Big).$$

We know that for $x \in [0,1]$, one has: $x(1-x) \in [0,1]$. Hence, it follows that:

$$w_t(x,t) - w_{xx}(x,t) \leq 1$$

for all $0 \leq x \leq 1, t > 0$. In particular, we deduce that:

$$\begin{cases} w_t(x,t) - w_{xx}(x,t) = 1, \text{ for } 0 < x < 1, t > 0 \\ w(x,0) = 0, \text{ for } 0 \leq x \leq 1 \\ w(0,t) = w(1,t) = 0, \text{ for } t > 0. \end{cases}$$

By using the comparison principle, it follows that, for all $x \in [0,1], t > 0$, the following holds:

$$u(x,t) \geq w(x,t) = \frac{1}{2} \cdot (1 - e^{-2t}) \cdot x(1-x) = (1 - e^{-2t})v(x).$$

d) Combining the results of parts b) and c), it follows that, for all $x \in [0,1], t > 0$, it holds that:

$$(1 - e^{-2t})v(x) \leq u(x,t) \leq v(x).$$

Letting $t \to \infty$, it follows that:

$$u(x,t) \to v(x)$$

as $t \to \infty$. $\square$

Exercise 2. *a) Find the function u solving (1) of the previous exercise by using separation of variables. Leave the Fourier coefficients in the form of an integral. [HINT: Consider the function $w := u - v$ for u, v as in the previous exercise.]*

b) Show that this is the unique solution of the problem (1).

c) By using the formula from part a), give an alternative proof of the fact that $u(x,t) \to v(x)$ as $t \to \infty$. In this part, one is allowed to assume that the Fourier coefficients at time zero are absolutely summable without proof.

Solution:

a) Let $\tilde{u}(x,t) := u(x,t) - \frac{1}{2}x(1-x)$. Then the function $\tilde{u}$ solves:

$$\begin{cases} \tilde{u}_t(x,t) - \tilde{u}_{xx}(x,t) = 0, \text{ for } 0 < x < 1, t > 0 \\ \tilde{u}(x,0) = -\frac{1}{2}x(1-x), \text{ for } 0 \leq x \leq 1 \\ \tilde{u}(0,t) = \tilde{u}(1,t) = 0, \text{ for } t > 0. \end{cases}$$

We look for $\tilde{u}$ in the form of a Fourier sine series with coefficients which depend on t.

$$\tilde{u}(x,t) = \sum_{n=1}^{\infty} A_n(t) \sin(n\pi x).$$

We first set $t = 0$ to deduce that:

$$\tilde{u}(x,0) = -\frac{1}{2}x(1-x) = \sum_{n=1}^{\infty} A_n(0)\sin(n\pi x) = -\frac{1}{2}x(1-x).$$

Hence, $A_n(0)$ equals the n-th Fourier sine series coefficient of the function $-\frac{1}{2}x(1-x)$ on $[0,1]$. In particular,

$$A_n(0) = 2\int_0^1 \Big(-\frac{1}{2}x(1-x)\Big)\sin(n\pi x)\,dx.$$

In order for $\tilde{u}$ to solve the heat equation, we need:

$$A_n'(t) - n^2\pi^2 A_n(t) = 0.$$

Hence:

$$A_n(t) = A_n(0)\cdot e^{-n^2\pi^2 t}.$$

Consequently:

$$\tilde{u}(x,t) = \sum_{n=1}^{\infty} A_n(0)\cdot e^{-n^2\pi^2 t}\cdot \sin(n\pi x).$$

We then deduce that:

$$u(x,t) = \frac{1}{2}x(1-x) + \sum_{n=1}^{\infty} A_n(0)\cdot e^{-n^2\pi^2 t}\cdot \sin(n\pi x).$$

b) Uniqueness of the problem (1) was shown in class by using the maximum principle and by using the energy method.

c) We note that:

$$|u(x,t) - v(x)| = \Big|\sum_{n=1}^{\infty} A_n(0)\cdot e^{-n^2\pi^2 t}\cdot \sin(n\pi x)\Big| \leq \sum_{n=1}^{\infty} |A_n(0)|\cdot e^{-n^2\pi^2 t} \leq e^{-\pi^2 t}\cdot \sum_{n=1}^{\infty}|A_n(0)|.$$

As is noted in the problem, we are allowed to assume that [1]

$$\sum_{n=1}^{\infty}|A_n(0)| < \infty.$$

The claim now follows. □

Exercise 3. *Suppose that $u : \mathbb{R}^3 \to \mathbb{R}$ is a harmonic function.*

a) By using the Mean Value Property (in terms of averages over spheres), show that, for all $x \in \mathbb{R}^3$, and for all $R > 0$, one has:

$$u(x) = \frac{3}{4\pi R^3}\int_{B(x,R)} u(y)\,dy.$$

b) Suppose, moreover, that $\int_{\mathbb{R}^3}|u(y)|\,dy < \infty$. Show that then, one necessarily obtains:

$$u(x) = 0$$

for all $x \in \mathbb{R}^3$.

[1] We can integrate by parts twice in the definition of $A_n(0)$ and use the fact that $-\frac{1}{2}x(1-x)$ vanishes at $x = 0$ and $x = 1$ in order to deduce that: $|A_n(0)| \leq \frac{C}{n^2}$ from where it indeed follows that $\sum_{n=1}^{\infty}|A_n(0)| < \infty$.

Solution:

a) Let us fix $x \in \mathbb{R}^3$. The Mean Value Property, proved in Exercise 1 of Homework Assignment 7, implies that, for all $r > 0$:

$$u(x) = \frac{1}{4\pi r^2} \int_{\partial B(x,r)} u(y)\, dS(y). \tag{2}$$

We note that:

$$\frac{3}{4\pi R^3} \int_{B(x,R)} u(y)\, dS(y) = \frac{3}{4\pi R^3} \int_0^R \Big(\int_{\partial B(x,r)} u(y)\, dS(y) \Big)\, dr.$$

By the Mean Value Property (2), it follows that this expression equals:

$$\frac{3}{4\pi R^3} \int_0^R 4\pi r^2 u(x)\, dr = u(x) \cdot \frac{3}{4\pi R^3} \cdot \int_0^R 4\pi r^2\, dr = u(x).$$

b) We note that, by part a), it follows that:

$$|u(x)| \leq \frac{3}{4\pi R^3} \int_{B(x,R)} |u(y)|\, dy \leq \frac{3}{4\pi R^3} \int_{\mathbb{R}^3} |u(y)|\, dy.$$

Since $\int_{\mathbb{R}^3} |u(y)|\, dy < \infty$, we can let $R \to \infty$ to deduce that $|u(x)| = 0$. It follows that u is identically equal to zero. $\square$

Exercise 4. *Suppose that $u : B(0,2) \to \mathbb{R}$ is a harmonic function on the open ball $B(0,2) \subseteq \mathbb{R}^2$, which is continuous on its closure $\overline{B(0,2)}$. Suppose that, in polar coordinates:*

$$u(2,\theta) = 3 \sin 5\theta + 1$$

for all $\theta \in [0, 2\pi]$.
a) Find the maximum and minimum value of u in $\overline{B(0,2)}$ without explicitly solving the Laplace equation.

b) Calculate $u(0)$ without explicitly solving the Laplace equation.

Solution:

a) By using the weak maximum principle for solutions to the Laplace equation, we know that the maximum of the function u on $\overline{B(0,2)}$ is achieved on $\partial B(0,2)$. We observe that the function $u(2,\theta) = 3\sin 5\theta + 1$ takes values in $[-2, 4]$. It equals -2 when $\sin 5\theta = -1$, which happens at $\theta = \frac{3\pi}{10}$ (for example). Moreover $u(2,\theta) = 4$ when $\sin 5\theta = 1$, which happens at $\theta = \frac{\pi}{10}$ (for example). Hence, the **maximum value** of u on $\overline{B(0,2)}$ is **4** and the **minimum value** of u on $\overline{B(0,2)}$ is $\mathbf{-2}$.

b) We use the Mean Value Property to deduce that $u(0)$ equals the average of u over the circle $\partial B(0,2)$. Since the average of the $3\sin 5\theta$ term equals zero, it follows that $u(0) = 1$. $\square$