MATH 425, PRACTICE MIDTERM EXAM 2.

Exercise 1. *Suppose that u solves the boundary value problem:*

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = 1, \ \textit{for } 0 < x < 1, t > 0 \\ u(x,0) = 0, \ \textit{for } 0 \leq x \leq 1 \\ u(0,t) = u(1,t) = 0, \ \textit{for } t > 0. \end{cases} \tag{1}$$

a) Find a function $v = v(x)$ which solves:

$$\begin{cases} -v_{xx}(x) = 1, \ \textit{for } 0 < x < 1 \\ v(0) = v(1) = 0. \end{cases}$$

b) Show that:

$$u(x,t) \leq v(x)$$

for all $x \in [0,1], t > 0$.

c) Show that:

$$u(x,t) \geq (1 - e^{-2t})v(x)$$

for all $x \in [0,1], t > 0$.

d) Deduce that, for all $x \in [0,1]$:

$$u(x,t) \to v(x)$$

as $t \to \infty$.

Exercise 2. *a) Find the function u solving* (1) *of the previous exercise by using separation of variables. Leave the Fourier coefficients in the form of an integral. [HINT: Consider the function $w := u - v$ for u, v as in the previous exercise.]*

b) Show that this is the unique solution of the problem (1).

c) By using the formula from part a), give an alternative proof of the fact that $u(x,t) \to v(x)$ as $t \to \infty$. In this part, one is allowed to assume that the Fourier coefficients at time zero are absolutely summable without proof.

Exercise 3. *Suppose that $u : \mathbb{R}^3 \to \mathbb{R}$ is a harmonic function.*

a) By using the Mean Value Property (in terms of averages over spheres), show that, for all $x \in \mathbb{R}^3$, and for all $R > 0$, one has:

$$u(x) = \frac{3}{4\pi R^3} \int_{B(x,R)} u(y)\, dy.$$

b) Suppose, moreover, that $\int_{\mathbb{R}^3} |u(y)|\, dy < \infty$. Show that then, one necessarily obtains:

$$u(x) = 0$$

for all $x \in \mathbb{R}^3$.

Exercise 4. *Suppose that* $u : B(0,2) \to \mathbb{R}$ *is a harmonic function on the open ball* $B(0,2) \subseteq \mathbb{R}^2$*, which is continuous on its closure* $\overline{B(0,2)}$*. Suppose that, in polar coordinates:*

$$u(2,\theta) = 3\sin 5\theta + 1$$

for all $\theta \in [0, 2\pi]$*.*
a) Find the maximum and minimum value of u *in* $\overline{B(0,2)}$ *without explicitly solving the Laplace equation.*

b) Calculate $u(0)$ *without explicitly solving the Laplace equation.*