PRACTICE HOMEWORK FOR MATH 425

The point of this problem set is just to review some facts from Calculus, Complex Analysis and from Ordinary Differential Equations. It is not meant to be turned in. Collaboration is encouraged.

Exercise 1. *Evaluate the integral:*

$$\int_0^{2\pi} e^{\theta} \sin\theta \, d\theta$$

a) by using Integration by parts.
b) by using complex numbers.

Let us recall **Euler's formula**:

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

for $\theta \in \mathbb{R}$.

Exercise 2. *Using Euler's formula, rederive the identities:*
a) $\sin(x+y) = \sin x \cos y + \cos x \sin y$.
b) $\cos(x+y) = \cos x \cos y - \sin x \sin y$.

Exercise 3. *Find all complex numbers z such that:*
a) $z^6 = 1$.
b) $z^7 = i$.
c) $Re(e^z) > 0$.

Exercise 4. *a) For what $c \in \mathbb{R}$ does there exist a non-zero function $w : [0, 2\pi] \to \mathbb{C}$ such that:*

$$w'' - c^2 w = 0$$

and such that $w(0) = w(2\pi) = 0$?
b) What if w instead solves $w'' + c^2 w = 0$ (again with the assumption that $w(0) = w(2\pi) = 0$)?

Exercise 5. *Suppose that $w : [0, +\infty) \to \mathbb{R}$ solves the ODE:*

$$aw'' + bw' + cw = 0 \tag{1}$$

for some constants a, b, c. Furthermore, we assume that $b \geq 0$.
a) Let us define the ***Energy*** *to be:*

$$E(t) := \frac{1}{2}\big[a(w'(t))^2 + c(w(t))^2\big].$$

Without solving the ODE (1), show that $E'(t) \leq 0$.
b) Under the additional assumption that $a > 0$ and $c > 0$, show that $w(0) = 0$ and $w'(0) = 0$ implies that $w(t) = 0$ for all $t \geq 0$.
c) Assume again that $a > 0$ and $c > 0$. Show that if w_1 and w_2 solve the ODE (1) and if $w_1(0) = w_2(0), w_1'(0) = w_2'(0)$, then one can deduce that $w_1(t) = w_2(t)$ for all $t \geq 0$. In this way, we obtain uniqueness of solutions to (1).