**MATH 425, MIDTERM EXAM 2, SOLUTIONS.**

Each exercise is worth 25 points.

**Exercise 1.** *Consider the initial value problem:*

$$\begin{cases} u_t - u_{xx} = 0, \text{ for } 0 < x < 1, \ t > 0 \\ u(x,0) = x(1-x), \text{ for } 0 \leq x \leq 1 \\ u(0,t) = 0, \ u(1,t) = 0, \text{ for } t > 0. \end{cases} \tag{1}$$

*a) Find the maximum of the function* $u$ *on* $[0,1]_x \times [0,+\infty)_t$.

*b) Show that, for all* $0 \leq x \leq 1, t \geq 0$ :

$$u(x,t) \geq 0.$$

*c) Show that, for all* $0 \leq x \leq 1, t \geq 0$:

$$u(x,t) \leq x(1-x)e^{-8t}.$$

*d) Given* $x \in [0,1]$, *calculate* $\lim_{t \to \infty} u(x,t)$.

**Solution:**

a) We observe that that the function $u$ equals zero on the lateral sides $x = 0$ and $x = 1$. Hence, by the Maximum Principle, it has to achieve its maximum on the bottom side $t = 0$. The function $x(1-x)$ achieves its maximum $\frac{1}{4}$ at $x = \frac{1}{2}$. Hence, the maximum of $u$ equals $\frac{1}{4}$ and it is achieved at the point $(x,t) = (\frac{1}{2}, 0)$.

b) **First solution:** We apply the *Minimum Principle*. We note by (2) that the function $u$ is non-negative on the lateral sides ($x = 0$ and $x = 1$) and on the bottom side ($t = 0$) of the infinite rectangle $[0,1]_x \times [0,+\infty)_t$. The claim then follows from the minimum principle. Strictly speaking, we should apply the Minimum Principle stated in class on a finite rectangle $[0,1]_x \times [0,T]_t$ and we then let $T \to +\infty$.

**Second solution:** We can apply the *Comparison Principle*. We recall the *Comparison Principle*, which was proved in Exercise 3 of Homework Assignment 4. We can summarize this principle as follows:

Suppose that:

$$\begin{cases} v_t - v_{xx} \geq w_t - x_{xx}, \text{ for } 0 < x < 1, \ t > 0 \\ v(x,0) \geq w(x,0), \text{ for } 0 \leq x \leq 1 \\ v(0,t) \geq w(0,t), \ v(1,t) \geq w(1,t), \text{ for } t > 0. \end{cases} \tag{2}$$

Then:

$$v(x,t) \geq w(x,t)$$

for all $x \in [0,1]$, $t > 0$. In other words, if $v_t - v_{xx} \geq w_t - w_{xx}$ and if $v \geq w$ on the bottom and lateral sides of $[0,1]_x \times [0,+\infty)_t$, then we can deduce that $v \geq w$ on all of $[0,1]_x \times [0,+\infty)_t$.

We now apply the Comparison Principle. Let us note $u = 0$ on the lateral sides and since $u$ equals $x(1-x)$, which is non-negative, on the bottom side. Hence, we can apply the Comparison Principle with $v = u$ and with $w = 0$ in order to deduce the claim.

c) In part c), we will have to apply the Comparison Principle.
Let us take:

$$v(x,t) := x(1-x)e^{-8t}.$$

We compute:

$$v_t(x,t) = -8x(1-x)e^{-8t}$$

and

$$v_{xx}(x,t) = -2e^{-8t}.$$

Hence:

$$v_t(x,t) - v_{xx}(x,t) = -8x(1-x)e^{-8t} + 2e^{-8t} = 2(1-4x(1-x))e^{-8t}.$$

Let us recall that we are considering $x \in [0,1]$ and so:

$$1 - 4x(1-x) \geq 1 - 4 \cdot \frac{1}{4} = 0,$$

since $x \mapsto x(1-x)$ achieves its maximum on $[0,1]$ at the point $x = \frac{1}{2}$. Hence:

$$v_t - v_{xx} \geq 0.$$

Let us also note that:

$$v(x,0) = u(x,0) = x(1-x)$$

for all $x \in [0,1]$.
Moreover,

$$v(0,t) = v(1,t) = u(0,t) = u(1,t) = 0$$

for all $t > 0$. It follows that we can apply the Comparison Principle with $v = x(1-x)e^{-8t}$ as above and with $w = u$, the solution to (2) in order to deduce the claim.

d) Let us fix $x \in [0,1]$. From parts *b*) and *c*), it follows that, for all $t > 0$:

$$0 \leq u(x,t) \leq x(1-x)e^{-8t}.$$

It follows that the limit as $t \to \infty$ of $u(x,t)$ equals zero. □

**Exercise 2.** *a) Find a solution to the following boundary value problem by separation of variables:*

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = \sin(5\pi x), \ \textit{for } 0 < x < 1, t > 0 \\ u(x,0) = 0, \ \textit{for } 0 \leq x \leq 1 \\ u(0,t) = u(1,t) = 0, \ \textit{for } t > 0. \end{cases} \tag{3}$$

*b) Is this the only solution to* (3)*?*

**Solution:**

a) We look for a solution of the form:

$$u(x,t) = A(t) \cdot \sin(5\pi x). \tag{4}$$

The reason why we look for such a solution is that the right-hand side of the equation contains a $\sin(5\pi x)$ term. We expect that this is the only frequency that will be present in the solution. In the form of $u$ that we are looking for, for each fixed $t$, the function $u(x,t)$ has a Fourier sine expansion in terms of $\sin(5\pi x)$. The coefficient will be a function of $t$.

Let us note that, for $u$ defined as in (4), the boundary conditions $u(0,t) = u(1,t) = 0$ are satisfied since $\sin(0) = \sin(5\pi) = 0$.

Our goal is to choose $A(t)$ such that $u$ solves the inhomogeneous heat equation. We compute:

$$u_t - u_{xx} = \Big\{ A'(t) + 25\pi^2 A(t) \Big\} \cdot \sin(5\pi x)$$

which, by the equation, equals:

$$\sin(5\pi x).$$

We can now equate the coefficient of $\sin(5\pi x)$ to deduce:

$$A'(t) + 25\pi^2 A(t) = 1. \tag{5}$$

Hence, the condition (5) guarantees that the function $u$ defined in (4) solves the PDE.

We now need to solve for $A(t)$. By the condition that $u(x,0) = A(0) \cdot \sin(5\pi x)$, it follows that $A(0) = 0$. Hence, we need to solve the following initial value problem to determine $A(t)$:

$$\begin{cases} A'(t) + 25\pi^2 A(t) = 1 \\ A(0) = 0. \end{cases}$$

We solve the ODE by multiplying with the integrating factor $e^{25\pi^2 t}$. The ODE then becomes:

$$e^{25\pi^2 t} A'(t) + 25\pi^2 e^{25\pi^2 t} A(t) = e^{25\pi^2 t}$$

i.e.

$$(e^{25\pi^2 t} A(t))' = e^{25\pi^2 t}.$$

Hence:

$$e^{25\pi^2 t} A(t) = A_0 + \frac{1}{25\pi^2} e^{25\pi^2 t}.$$

We note that $A(0) = 0$ implies that $A_0 = -\frac{1}{25\pi^2}$. Consequently:

$$A(t) = \frac{1}{25\pi^2} \cdot \Big\{ 1 - e^{-25\pi^2 t} \Big\}.$$

It follows that:

$$u(x,t) = \frac{1}{25\pi^2} \cdot \Big\{ 1 - e^{-25\pi^2 t} \Big\} \cdot \sin(5\pi x).$$

b) We know from class that the boundary value problem for the heat equation on a spatial interval of finite length admits unique solutions, either by applying the Maximum Principle or by applying the Energy Method. Hence, the function $u$ from part a) is the unique solution to (3). □

**Exercise 3.** *Let us recall that a function $u : \mathbb{R}^n \to \mathbb{R}$ is called subharmonic if $\Delta u \geq 0$. In particular, every harmonic function is subharmonic.*

*a) Given a harmonic function $u : \mathbb{R}^n \to \mathbb{R}$, show that the function $v := u^2$ is subharmonic on $\mathbb{R}^n$.*
*b) Under which conditions on $u$ can we deduce that the function $v$ defined above is harmonic?*

**Solution:**

a) We compute, for $1 \leq j \leq n$:

$$v_{v_j} = (u^2)_{x_j} = 2uu_{x_j}$$

and so:

$$v_{x_j x_j} = (u^2)_{x_j x_j} = 2u_{x_j} u_{x_j} + 2uu_{x_j x_j} = 2u_{x_j}^2 + 2uu_{x_j x_j}$$

We sum in $j = 1, \ldots, n$ in order to deduce:

$$\Delta v = 2\sum_{j=1}^{n} u_{x_j}^2 + 2u\Delta u = 2|\nabla u|^2 + 2u\Delta u.$$

Since $\Delta u = 0$, this quantity equals: $2|\nabla u|^2$ which is non-negative. Hence, $v$ is subharmonic.

b) From part a), we recall that:

$$\Delta v = 2|\nabla u|^2.$$

In particular $v$ is harmonic if and only if $\nabla u = 0$, which is the case if and only if $u$ is constant. □

**Exercise 4.** *Suppose that* $u : B(0,1) \to \mathbb{R}$ *is a harmonic function on the open ball* $B(0,1) \subseteq \mathbb{R}^2$*, which extends to a continuous function on its closure* $\overline{B(0,1)}$*.*

*Suppose that, in polar coordinates:*

$$u(1,\theta) = 2 + 3\sin\theta$$

*for all* $\theta \in [0, 2\pi]$*.*

*a) Find the minimum and the maximum of* $u$ *on* $\overline{B(0,1)}$*.*

*b) Find the value of* $u$ *at the origin.*

*c) Find an expression for the value of* $u$ *at the point* $(\frac{1}{2}, \frac{\pi}{2})$ *in polar coordinates by using Poisson's formula. Don't explicitly evaluate the integral.*

*d) Does there exist a point in* $B(0,1)$ *at which* $u$ *takes the value* $5$*?*

**Solution:**

a) We use the Weak Maximum Principle for the Laplace equation in order to deduce that $u$ achieves its maximum and minimum on the boundary. More precisely:

$$\min_{\overline{B(0,1)}} u = \min_{\partial B(0,1)} u$$

and

$$\max_{\overline{B(0,1)}} u = \max_{\partial B(0,1)} u.$$

We know that for all $\theta \in [0, 2\pi]$:

$$-1 \leq 2 + 3\sin\theta \leq 5.$$

Moreover:

$$2 + 3\sin\Big(\frac{3\pi}{2}\Big) = -1$$

and

$$2 + 3\sin\Big(\frac{\pi}{2}\Big) = 5.$$

Hence:

$$\min_{\overline{B(0,1)}} u = \min_{\partial B(0,1)} u = -1$$

and

$$\max_{\overline{B(0,1)}} u = \max_{\partial B(0,1)} u = 5.$$

b) We can use the Mean Value Property to deduce that the value of $u$ at the origin equals the average of the function $u$ on the circle $\partial B(0,1)$. In particular:

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} (2 + 3\sin\theta)\,d\theta = 2,$$

since $\int_0^{2\pi} \sin\theta\,d\theta = 0$.

c) We use Poisson's formula and we compute:

$$u\Big(\frac{1}{2}, \frac{\pi}{2}\Big) = \frac{1 - (\frac{1}{2})^2}{2\pi} \cdot \int_0^{2\pi} \frac{2 + 3\sin\phi}{(\frac{1}{2})^2 - 2\cdot\frac{1}{2}\cdot 1\cos(\frac{\pi}{2} - \phi) + 1}\,d\phi = \frac{3}{2\pi}\cdot\int_0^{2\pi}\frac{2 + 3\sin\phi}{5 - 4\sin\phi}\,d\phi.$$

d) Suppose that there were a point $x_0 \in B(0,1)$ at which $u(x_0) = 5$, then by part a), it would follow that:

$$u(x_0) = \max_{\overline{B(0,1)}} u.$$

Hence, $u$ achieves its maximum at an interior point. The Strong Maximum Principle would then imply that $u$ was constant. However, $u$ is not constant on the boundary $\partial B(0,1)$, which gives us a contradiction. Hence, there is no such point $x_0$ in the interior. $\square$