MATH 425, MIDTERM EXAM 2. APRIL 9, 2013.

No books, notes or calculators are allowed during the test. There are four exercises. Each exercise is worth 25 points. In your solutions, you are allowed to use any fact that was proved in class or on the homework, provided that it is clearly stated. Good Luck!

Exercise 1. *Consider the initial value problem:*

$$\begin{cases} u_t - u_{xx} = 0, \text{ for } 0 < x < 1, \ t > 0 \\ u(x,0) = x(1-x), \text{ for } 0 \leq x \leq 1 \\ u(0,t) = 0, \ u(1,t) = 0, \text{ for } t > 0. \end{cases} \tag{1}$$

a) Find the maximum of the function u *on* $[0,1]_x \times [0,+\infty)_t$.

b) Show that, for all $0 \leq x \leq 1, t \geq 0$:

$$u(x,t) \geq 0.$$

c) Show that, for all $0 \leq x \leq 1, t \geq 0$:

$$u(x,t) \leq x(1-x)e^{-8t}.$$

d) Given $x \in [0,1]$, *calculate* $\lim_{t \to \infty} u(x,t)$.

Exercise 2. *a) Find a solution to the following boundary value problem by separation of variables:*

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = \sin(5\pi x), \text{ for } 0 < x < 1, t > 0 \\ u(x,0) = 0, \text{ for } 0 \leq x \leq 1 \\ u(0,t) = u(1,t) = 0, \text{ for } t > 0. \end{cases} \tag{2}$$

b) Is this the only solution to (2)*?*

Exercise 3. *Let us recall that a function* $u : \mathbb{R}^n \to \mathbb{R}$ *is called subharmonic if* $\Delta u \geq 0$. *In particular, every harmonic function is subharmonic.*

a) Given a harmonic function $u : \mathbb{R}^n \to \mathbb{R}$, *show that the function* $v := u^2$ *is subharmonic on* $\mathbb{R}^n$.

b) Under which conditions on u *can we deduce that the function* v *defined above is harmonic?*

Exercise 4. *Suppose that* $u : B(0,1) \to \mathbb{R}$ *is a harmonic function on the open ball* $B(0,1) \subseteq \mathbb{R}^2$, *which extends to a continuous function on its closure* $\overline{B(0,1)}$.

Suppose that, in polar coordinates:

$$u(1,\theta) = 2 + 3\sin\theta$$

for all $\theta \in [0, 2\pi]$.

a) Find the minimum and the maximum of u *on* $\overline{B(0,1)}$.

b) Find the value of u *at the origin.*

c) Find an expression for the value of u *at the point* $(\frac{1}{2}, \frac{\pi}{2})$ *in polar coordinates by using Poisson's formula. Don't explicitly evaluate the integral.*

d) Does there exist a point in $B(0,1)$ *at which* u *takes the value* 5*?*