MATH 425, MIDTERM EXAM 1 SOLUTIONS.

Exercise 1. *(20 points)*
In this exercise, the functions $u = u(x_1, x_2)$ are assumed to depend on two variables x_1, x_2. Determine, with an explanation, which of the following operators are linear:
a) $\mathcal{L}_1(u) := \sin(x_1) \cdot \frac{\partial^2 u}{\partial x_1^2} + \cos(x_2) \cdot \frac{\partial^2 u}{\partial x_2^2}$
b) $\mathcal{L}_2(u) := \Delta u + x_1$

Solution:

a) The operator $\mathcal{L}_1$ is linear. Namely, given functions $u = u(x_1, x_2), v = v(x_1, x_2)$ and scalars a, b, we note that:

$$\mathcal{L}_1(au + bv) = \sin(x_1) \cdot \frac{\partial^2}{\partial x_1^2}\big(au + bv\big) + \cos(x_2) \cdot \frac{\partial^2}{\partial x_2^2}\big(au + bv\big) =$$

$$= \sin(x_1) \cdot \Big(a\frac{\partial^2 u}{\partial x_1^2} + b\frac{\partial^2 v}{\partial x_1^2}\Big) + \cos(x_2) \cdot \Big(a\frac{\partial^2 u}{\partial x_2^2} + b\frac{\partial^2 v}{\partial x_2^2}\Big) =$$

$$= a \cdot \Big(\sin(x_1) \cdot \frac{\partial^2 u}{\partial x_1^2} + \cos(x_2) \cdot \frac{\partial^2 u}{\partial x_2^2}\Big) + b \cdot \Big(\sin(x_1) \cdot \frac{\partial^2 v}{\partial x_1^2} + \cos(x_2) \cdot \frac{\partial^2 v}{\partial x_2^2}\Big) =$$

$$= a\mathcal{L}_1(u) + b\mathcal{L}_2(v).$$

b) The operator $\mathcal{L}_2$ isn't linear. Namely, we note that $\mathcal{L}_2(0) = x_1$, which is not identically zero. However, we recall that for all linear operators T, it is necessarily the case that $T(0) = 0$. Hence, $\mathcal{L}_2$ isn't linear indeed. $\square$

Exercise 2. *(30 points)*
Suppose that $c > 0$ is given. Consider the PDE:

$$u_t + cu_x = 0, \text{ for } x \in \mathbb{R},\ t \in \mathbb{R}.$$

a) Write the general solution to the PDE and check that it solves the equation.
b) Show that the general solution to the PDE is given by the answer you found in part a).
c) Give a one-sentence description of a physical phenomenon which is modeled by this PDE. (You don't need to explain the physical derivation of the equation!).

Solution:

a) The general solution is given by $u(x,t) = f(x - ct)$ for some (differentiable) function f. We note that, for u defined as above, one obtains: $u_t(x,t) = -cf'(x - ct)$ and $u_x(x,t) = f'(x - ct)$. Consequently:

$$u_t(x,t) + cu_x(x,t) = -cf'(x - ct) + cf'(x - ct) = 0.$$

Hence, u solves the PDE.

b) We look at the directional derivative of the function $u = u(x,t)$ in the direction of the vector $(1, c)$. Here, the first component refers to the x-variable and the second component refers to the t-variable. In other words, we fix $(x_0, t_0) \in \mathbb{R} \times \mathbb{R}$ and we use the Chain Rule in order to compute:

$$\frac{d}{dp}u(x_0 + p, t_0 + cp) = u_x(x_0 + p, t_0 + cp) + cu_x(x_0 + p, t_0 + cp) = 0.$$

In other words, we deduce that, for all $p \in \mathbb{R}$, one has:

$$u(x_0 + p, t_0 + cp) = u(x_0, t_0).$$

Let us note, as p varies over $\mathbb{R}$, the points $(x_0 + p, t_0 + cp)$ define a line in the xt-plane which passes through (x_0, t_0) and has slope c. The equation of this line is given by:

$$x - ct = x_0 - ct_0.$$

By the above argument, it follows that u is constant along each line in the xt-plane with slope c. Equivalently, the function u just depends on the quantity $x - ct$ (because this quantity determines on which of the lines of slope c the point (x,t) lies).

c) A possible physical interpretation is the following: $u(x,t)$ denotes the concentration in grams per centimeter at time t of ink being transported in the horizontal direction through a pipe full of water to the right at the rate of c grams per second. In this model, we are ignoring diffusion effects. (Since the transport is only in the horizontal direction, we only need to look at how the concentration varies in the x-direction; this interpretation makes sense in one, two or three dimensions). □

Exercise 3. *(30 points)*
Consider the initial value problem:

$$\begin{cases} u_t - ku_{xx} = f, \text{ for } x \in \mathbb{R}, t > 0 \\ u(x,0) = \phi. \end{cases}$$

Here $k > 0$ is a constant, $\phi = \phi(x)$ is a function of x and $f = f(x,t)$ is a function of x and t.

a) Write down a solution to this initial value problem, expressing the heat kernel explicitly in terms of exponentials.
b) Suppose that $\phi \geq 0$ and $\phi(x) = 1$ whenever $x \in [0,1]$. Furthermore, suppose that $f \geq 0$. Show that:

$$u(x,t) > 0 \text{ for all } x \in \mathbb{R}, t > 0.$$

[Partial credit will be given for showing the claim in the special case when $f = 0$.]

c) Write down a solution to the initial value problem:

$$\begin{cases} u_t - ku_{xx} + Au = f, \text{ for } x \in \mathbb{R}, t > 0 \\ u(x,0) = \phi. \end{cases}$$

where now $A \in \mathbb{R}$ is a constant. As in part a), express your solution explicitly in terms of exponentials. [HINT: Multiply the equation with e^{At}.]

Solution:

a) We will use Duhamel's Principle to write:

$$u(x,t) = \int_{-\infty}^{+\infty} S(x-y,t) \cdot \phi(y)\, dy + \int_0^t \int_{-\infty}^{+\infty} S(x-y,t-s) \cdot f(y,s)\, dyds$$

where $S(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}$ is the heat kernel.

Hence, we can explicitly write:

$$u(x,t) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-y)^2}{4kt}} \cdot \phi(y)\, dy + \int_0^t \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi k(t-s)}} e^{-\frac{(x-y)^2}{4k(t-s)}} \cdot f(y,s)\, dyds. \tag{1}$$

b) Since $\phi \geq 0$, the integrand inside the first integral in (1), i.e. $\frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-y)^2}{4kt}} \cdot \phi(y)$ is non-negative, so restricting the domain of integration in the y variable from $(-\infty, +\infty)$ to $[0,1]$ will not increase the value of the integral. Moreover, since $f \geq 0$, the second integral in (1), i.e. $\int_0^t \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi k(t-s)}} e^{-\frac{(x-y)^2}{4k(t-s)}} \cdot f(y,s)\, dyds$ is non-negative. Hence, for all $x \in \mathbb{R}, t > 0$, one obtains:

$$u(x,t) \geq \int_0^1 \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-y)^2}{4kt}} \cdot \phi(y)\, dy = \int_0^1 \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-y)^2}{4kt}}\, dy > 0.$$

In the above line, we also used the assumption that $\phi(y) = 1$ for all $y \in [0,1]$.

c) Let us multiply both sides of the equation by e^{At} (We note that this is a good idea by observing that the homogeneous equation without the diffusion becomes $w_t + Aw = 0$.)

$$e^{At}u_t - ke^{At}u_{xx} + Aa^{At}u = e^{At}f(x,t).$$

We note that: $e^{At}u_t + Ae^{At}u = (e^{At}u)_t$. Hence, if we define $v(x,t) := e^{At} \cdot u(x,t)$, the function v will solve the following initial value problem:

$$\begin{cases} v_t - kv_{xx} = e^{At}f, \text{ for } x \in \mathbb{R}, t > 0 \\ v(x,0) = \phi. \end{cases}$$

Here, we used the fact that $e^{At}u_{xx} = (e^{At}u)_{xx} = v_{xx}$ and $v(x,0) = e^{A \cdot 0}u(x,0) = u(x,0) = \phi(x)$. In other words, v solves the inhomogeneous heat equation with a slightly different right-hand side source term. We can now use the formula from part *a*) to deduce that:

$$v(x,t) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-y)^2}{4kt}} \cdot \phi(y)\, dy + \int_0^t \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi k(t-s)}} e^{-\frac{(x-y)^2}{4k(t-s)}} \cdot e^{As} \cdot f(y,s)\, dyds.$$

Since $u(x,t) = e^{-At} \cdot v(x,t)$, it follows that:

$$u(x,t) = e^{-At} \cdot \Big(\int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-y)^2}{4kt}} \cdot \phi(y)\, dy + \int_0^t \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi k(t-s)}} e^{-\frac{(x-y)^2}{4k(t-s)}} \cdot e^{As} \cdot f(y,s)\, dyds \Big). \square$$

Exercise 4. *(20 points)*
In this exercise $u = u(x,y)$ is a function of two variables.
a) Solve the equation $yu_x + xu_y = 0$ with the initial condition $u(x,0) = x^3$.
b) In which region of the xy-plane is the solution uniquely determined?
c) If we were to additionally prescribe the values of u along the y-axis, would this uniquely determine u on the whole xy-plane? (We are implicitly assuming that we are prescribing the values of u on the y axis in such a way that the solution exists).

Solution:

a) We will apply the method of characteristics. We rewrite the PDE as:

$$u_x + \frac{x}{y}u_y = 0.$$

The ODE one has to solve is then:

$$\frac{dy}{dx} = \frac{x}{y}.$$

By separation of variables, it follows that:

$$x\, dx = y\, dy$$

The characteristic curves are given are given by the connected components of:

$$x^2 - y^2 = C. \tag{2}$$

for $C \in \mathbb{R}$.

Let us first solve the problem in full generality and we later substitute the value of $u(x, 0)$. One has to be a bit careful here; for $C \neq 0$, equation (2) gives us two segments of a hyperbola (so not one connected curve), and for $C = 0$, it gives us the union of the lines $y = x$ and $y = -x$. In any case, by the method of characteristics, **the function u will be constant on each of the connected components of these curves**.

It follows that:

$$u(x,y) = \begin{cases} C, \text{ if } y = \pm x \\ h_1(x^2 - y^2), \text{ if } x^2 - y^2 > 0 \text{ and } x > 0 \textbf{ (Rightwards facing hyperbolic segments)} \\ h_2(x^2 - y^2), \text{ if } x^2 - y^2 > 0 \text{ and } x < 0 \textbf{ (Leftwards facing hyperbolic segments)} \\ g_1(x^2 - y^2), \text{ if } x^2 - y^2 < 0 \text{ and } y > 0 \textbf{ (Upwards facing hyperbolic segments)} \\ g_2(x^2 - y^2), \text{ if } x^2 - y^2 < 0 \text{ and } y < 0 \textbf{ (Downwards facing hyperbolic segments)} \end{cases}$$

Let us now consider the concrete example where $u(x, 0) = x^3$. We note that $(x_0, 0)$ is the intersection of the x axis with the set $x^2 - y^2 = x_0^2$ in the half-plane where x has the same sign as x_0 (if $x_0 = 0$, this point is just $(0, 0)$).
In particular, we note that:

i) $C = u(0, 0) = 0$.
ii) $h_1(x^2) = x^3$ for $x > 0$.
iii) $h_2(x^2) = x^3$ for $x < 0$.

Thus, *ii)* and *iii)* imply that we can take $h_1(x) = |x|^{\frac{3}{2}}$ and $h_2(x) = -|x|^{\frac{3}{2}}$.

Consequently, we obtain:

$$u(x,y) = \begin{cases} 0, \text{ if } y = \pm x \\ (x^2 - y^2)^{\frac{3}{2}}, \text{ if } x^2 - y^2 > 0 \text{ and } x > 0 \textbf{ (Rightwards facing hyperbolic segments)} \\ -(x^2 - y^2)^{\frac{3}{2}}, \text{ if } x^2 - y^2 > 0 \text{ and } x < 0 \textbf{ (Leftwards facing hyperbolic segments)} \\ g_1(x^2 - y^2), \text{ if } x^2 - y^2 < 0 \text{ and } y > 0 \textbf{ (Upwards facing hyperbolic segments)} \\ g_2(x^2 - y^2), \text{ if } x^2 - y^2 < 0 \text{ and } y < 0 \textbf{ (Downwards facing hyperbolic segments)} \end{cases}$$

Again, we need to choose the functions g_1 and g_2 in such a way that the function u is differentiable.

b) Since the value of u is given on the x-axis, it follows that the solution is uniquely determined along the characteristic curves which intersect the x-axis. These includes the leftwards and rightwards facing hyperbolic segments as well as the union of the lines $y = x$ and $y = -x$. Hence, the solution u is uniquely determined on the set where $x^2 - y^2 \geq 0$.

c) If we were to prescribe the values of u along the y-axis, the solution would additionally be uniquely determined on the upwards and downwards facing hyperbolic segments, i.e. we could determine the functions g_1 and g_2. Consequently, the solution would be uniquely determined on the whole xy-plane. □