MATH 425, MIDTERM EXAM 1. FEBRUARY 12, 2013.

No books, notes or calculators are allowed during the test. There are four exercises. The maximal number of points is 100. Good Luck!

Exercise 1. *(20 points)*
In this exercise, the functions $u = u(x_1, x_2)$ are assumed to depend on two variables x_1, x_2. Determine, with an explanation, which of the following operators are linear:
a) $\mathcal{L}_1(u) := \sin(x_1) \cdot \frac{\partial^2 u}{\partial x_1^2} + \cos(x_2) \cdot \frac{\partial^2 u}{\partial x_2^2}$
b) $\mathcal{L}_2(u) := \Delta u + x_1$.

Exercise 2. *(30 points)*
Suppose that $c > 0$ is given. Consider the PDE:

$$u_t + cu_x = 0, \text{ for } x \in \mathbb{R},\, t \in \mathbb{R}.$$

a) Write down the general solution to the PDE and check that it solves the equation.
b) Prove that the general solution to the PDE is given by the answer you found in part a).
c) Give a brief description of a physical phenomenon which is modeled by this PDE. (You don't need to explain the details of the physical derivation of the equation!).

Exercise 3. *(30 points)*
Consider the initial value problem:

$$\begin{cases} u_t - ku_{xx} = f, \text{ for } x \in \mathbb{R},\, t > 0 \\ u(x,0) = \phi. \end{cases}$$

Here $k > 0$ is a constant, $\phi = \phi(x)$ is a function of x and $f = f(x,t)$ is a function of x and t.

a) Write down a solution to this initial value problem, expressing the heat kernel explicitly in terms of exponentials.
b) Suppose that $\phi \geq 0$ and $\phi(x) = 1$ whenever $x \in [0,1]$. Furthermore, suppose that $f \geq 0$. Show that:

$$u(x,t) > 0 \text{ for all } x \in \mathbb{R},\, t > 0.$$

[Partial credit will be given for showing the claim in the special case when $f = 0$.]

c) Write down a solution to the initial value problem:

$$\begin{cases} u_t - ku_{xx} + Au = f, \text{ for } x \in \mathbb{R}, t > 0 \\ u(x,0) = \phi. \end{cases}$$

where now $A \in \mathbb{R}$ is a constant. As in part a), express your solution explicitly in terms of exponentials. [HINT: Multiply the equation with e^{At}.]

Exercise 4. *(20 points)*
In this exercise $u = u(x,y)$ is a function of two variables.
a) Solve the equation $yu_x + xu_y = 0$ with the condition $u(x,0) = x^3$.
b) In which region of the xy-plane is the solution uniquely determined?
c) If we were to additionally prescribe the values of u along the y-axis, would this uniquely determine u on the whole xy-plane? (We are implicitly assuming that we are prescribing the values of u on the y axis in such a way that the solution exists).