MATH 425, HOMEWORK 8

Each problem is worth 10 points.

Exercise 1. *(Green's functions in two dimensions)*

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain. Suppose that $u : \Omega \to \mathbb{R}$ is a harmonic function which extends continuously to $\overline{\Omega} = \Omega \cup \partial\Omega$.

a) Prove that, for all $x_0 \in \Omega$:

$$u(x_0) = \frac{1}{2\pi} \int_{\partial\Omega} \Big[u(x) \cdot \frac{\partial}{\partial n} \log|x - x_0| - \frac{\partial u}{\partial n}(x) \cdot \log|x - x_0| \Big] \, ds.$$

Here, ds denotes the arclength element on $\partial\Omega$ (recall that each connected component of $\partial\Omega$ is a smooth curve).

b) Formulate a definition for the Green's function for the Laplace equation on the two-dimensional domain Ω as in part a).

c) Show that, for fixed $x_0 \in \Omega$, and for the right definition of the Green's function $G(x, x_0)$, it is true that:

$$u(x_0) = \int_{\partial\Omega} u(x) \cdot \frac{\partial G(x, x_0)}{\partial n} \, dS$$

for all harmonic functions u as in part a).

Solution:

a) We can apply translation by x_0 and see that it suffices to consider only the special case when Ω contains the origin and $x_0 = 0$.

Let us first show that, on $\mathbb{R}^2 \setminus \{0\}$, one has:

$$\Delta \log|x| = 0.$$

We write $\log|x|$ as $\log \sqrt{x_1^2 + x_2^2}$.
Hence, by the Chain Rule:

$$(\log|x|)_{x_1} = \frac{1}{\sqrt{x_1^2 + x_2^2}} \cdot \frac{2x_1}{2\sqrt{x_1^2 + x_2^2}} = \frac{x_1}{x_1^2 + x_2^2}.$$

$$(\log|x|)_{x_1 x_1} = \frac{1}{x_1^2 + x_2^2} - \frac{2x_1^2}{(x_1^2 + x_2^2)^2}.$$

By symmetry:

$$(\log|x|)_{x_2 x_2} = \frac{1}{x_1^2 + x_2^2} - \frac{2x_2^2}{(x_1^2 + x_2^2)^2}.$$

Summing the previous two identities, we obtain:

$$\Delta \log|x| = 0.$$

Alternatively, we can use the formula for Laplace's operator in polar coordinates:

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$$

in order to deduce that:

$$\Delta \log r = \Big(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \Big) \log r = -\frac{1}{r^2} + \frac{1}{r^2} = 0.$$

Let us now suppose that $\epsilon > 0$ is given and we consider the domain $\Omega_\epsilon := \Omega \setminus B(0,\epsilon)$. We can apply Green's second identity to deduce that:

$$\int_{\partial\Omega_\epsilon} \Big[u(x) \cdot \frac{\partial}{\partial n} \log|x| - \frac{\partial u}{\partial n}(x) \cdot \log|x|\Big]\, ds = \int_{\Omega_\epsilon} \Big[u(x) \cdot \Delta \log|x| - \log|x| \cdot \Delta u(x)\Big]\, dx = 0.$$

We note that:

$$\partial\Omega_\epsilon = \partial\Omega \cup \partial B(0,\epsilon).$$

Hence, it follows that:

$$\int_{\partial\Omega} \Big[u(x) \cdot \frac{\partial}{\partial n} \log|x| - \frac{\partial u}{\partial n}(x) \cdot \log|x|\Big]\, ds = -\int_{\partial B(0,\epsilon)} \Big[u(x) \cdot \frac{\partial}{\partial n} \log|x| - \frac{\partial u}{\partial n}(x) \cdot \log|x|\Big]\, ds$$

We need to show that the right-hand side converges to $2\pi u(0)$ as $\epsilon \to 0$. On $\partial B(0,\epsilon)$, we know that $\frac{\partial}{\partial n} = -\frac{\partial}{\partial r}$. Hence, the first term equals:

$$\frac{1}{\epsilon} \int_{\partial B(0,\epsilon)} u(x)\, ds$$

Since u is continuous, this quantity converges to $2\pi u(0)$ as $\epsilon \to 0$. The second term equals:

$$\log\epsilon \int_{\partial B(0,\epsilon)} \frac{\partial u}{\partial n}\, ds.$$

We can find $M > 0$, independent of ϵ such that $\left|\frac{\partial u}{\partial n}\right| \leq M$. Hence, we obtain:

$$\Big|\log\epsilon \int_{\partial B(0,\epsilon)} \frac{\partial u}{\partial n}\, ds\Big| \leq 2\pi M \cdot \epsilon \cdot |\log\epsilon|.$$

In order to prove the claim, we need to show that:

$$\lim_{\epsilon\to 0} \Big(\epsilon \cdot \log\epsilon\Big) = 0.$$

We note that this is not immediately obvious since, as $\epsilon \to 0$, the quantity $\log\epsilon \to -\infty$. Hence, the goal is to show that ϵ goes to zero faster than $\log\epsilon$ goes to $-\infty$. We can look at an example first to see why this should be true. Namely, if we take $\epsilon_n = \frac{1}{2^n}$, then $\epsilon_n \to 0$ as $n \to \infty$ and $\log\epsilon_n = -n\log 2$. Hence:

$$\epsilon_n \cdot \log\epsilon_n = -\frac{n\log 2}{2^n} \to 0$$

as $n \to \infty$. In order to treat the general case, we can use the L'Hôpital rule:

$$\lim_{x\to 0+} \Big(x \cdot \log x\Big) = \lim_{x\to 0+} \frac{\log x}{\frac{1}{x}} = \lim_{x\to 0+} \frac{(\log x)'}{\left(\frac{1}{x}\right)'} =$$

$$= \lim_{x\to 0+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0+} (-x) = 0.$$

Alternatively, we can look at the function $f(x) = -x \cdot \log x$. The function f is non-negative for $x \in (0,1]$. By the product rule:

$$f'(x) = -1 - \log x > 0$$

for all $x \in (0,\delta)$ when $\delta > 0$ is sufficiently small. Hence, f is monotonically increasing on $(0,\delta)$. From the earlier calculations, we know that $f(\frac{1}{2^n}) \to 0$ as $n \to \infty$, it follows that $\lim_{x\to 0+} f(x) = 0$. The claim now follows.

b) We can now define the Green's function for a two-dimensional domain Ω and $x_0 \in \Omega$ to be a function $G(\cdot, x_0) : \Omega \setminus \{x_0\} \to \mathbb{R}$ satisfying the following properties:

i) $G(x,x_0)$ is twice continuously differentiable on $\Omega \setminus \{x_0\}$. Moreover,

$$\Delta_x G(x,x_0) = 0 \text{ on } \Omega \setminus \{x_0\}.$$

ii) $G(x,x_0)$ for all $x \in \partial\Omega$.

iii) The function $G(x, x_0) - \frac{1}{2\pi} \log |x - x_0|$ is finite at x_0 and it is harmonic on all of Ω. (The choice of the term $-\frac{1}{2\pi} \log |x - x_0|$ will become clear in part c)).

c) Let us fix $x_0 \in \Omega$. Suppose that $G(x, x_0)$ is as in part b). We let

$$H(x, x_0) := G(x, x_0) - \frac{1}{2\pi} \log |x - x_0|.$$

Then $\Delta H = 0$ on Ω. We recall that $\Delta u = 0$ on Ω. Hence, we obtain, by Green's second identity, :

$$0 = \int_{\partial\Omega} \Big(u(x) \frac{\partial H(x, x_0)}{\partial n} - \frac{\partial u}{\partial n} H(x, x_0) \Big) \, ds.$$

We recall from part a) that:

$$u(x_0) = \frac{1}{2\pi} \int_{\partial\Omega} \Big[u(x) \cdot \frac{\partial}{\partial n} \log |x - x_0| - \frac{\partial u}{\partial n}(x) \cdot \log |x - x_0| \Big] \, ds.$$

We add the previous two identities to deduce that:

$$u(x_0) = \int_{\partial\Omega} \Big[u(x) \cdot \frac{\partial G(x, x_0)}{\partial n} - \frac{\partial u}{\partial n}(x) \cdot G(x, x_0) \Big] \, ds.$$

Since $G(x, x_0) = 0$ for $x \in \partial\Omega$, we obtain that:

$$u(x_0) = \int_{\partial\Omega} u(x) \cdot \frac{\partial G(x, x_0)}{\partial n} \, ds. \ \square$$

Exercise 2. *(An averaging property for smooth functions)*
Suppose that $\phi : \mathbb{R}^3 \to \mathbb{R}$ is a smooth function which equals zero outside of some ball centered at the origin.

a) Prove that:

$$\phi(0) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x|} \cdot \Delta\phi(x) \, dx.$$

b) Why is identity in part a) immediate if the function ϕ is assumed to be harmonic?

Solution:

a) Let us assume that $\phi = 0$ outside of $B(0, R) \subseteq \mathbb{R}^3$ and let $\epsilon > 0$ be given. We let:

$$\Omega_\epsilon := B(0, 2R) \setminus B(0, \epsilon).$$

Let us recall that, on $\mathbb{R}^3$, one has:

$$\Delta\Big(\frac{1}{|x|}\Big) = 0.$$

We now apply Green's second identity, noting that ϕ and $\frac{1}{|x|}$ are both smooth on Ω_ϵ in order to deduce that:

$$\int_{\Omega_\epsilon} \Big[\frac{1}{|x|} \cdot \Delta\phi(x) - \Delta\Big(\frac{1}{|x|}\Big) \cdot \phi(x) \Big] \, dx = \int_{\partial\Omega_\epsilon} \Big[\frac{1}{|x|} \cdot \frac{\partial\phi}{\partial n} - \frac{\partial}{\partial n}\Big(\frac{1}{|x|}\Big) \cdot \phi(x) \Big] \, dS(x).$$

We note that $\partial\Omega_\epsilon$ consists of two parts: $\partial B(0, \epsilon)$ and $\partial B(0, 2R)$. Since, by assumption, ϕ vanishes near $\partial B(0, 2R)$, it follows that the contribution to the right-hand side from the outer boundary $\partial B(0, 2R)$ equals to zero. Moreover, we know that $\Delta\Big(\frac{1}{|x|}\Big) = 0$ on Ω_ϵ. Hence, it follows that:

$$\int_{\Omega_\epsilon} \frac{1}{|x|} \cdot \Delta\phi(x) \, dx = \int_{\partial B(0,\epsilon)} \Big[\frac{1}{|x|} \cdot \frac{\partial\phi}{\partial n} - \frac{\partial}{\partial n}\Big(\frac{1}{|x|}\Big) \cdot \phi(x) \Big] \, dS(x).$$

We note that $\Delta\phi = 0$ for $|x| \geq 2R$ and we deduce that:

$$\int_{|x|\geq\epsilon} \frac{1}{|x|} \cdot \Delta\phi(x) \, dx = \int_{\partial B(0,\epsilon)} \Big[\frac{1}{|x|} \cdot \frac{\partial\phi}{\partial n} - \frac{\partial}{\partial n}\Big(\frac{1}{|x|}\Big) \cdot \phi(x) \Big] \, dS(x).$$

We now let $\epsilon \to 0$. Arguing analogously as in class, we note that the the right-hand side converges to $-4\pi\phi(0)$. Hence:

$$\int_{\mathbb{R}^3} \frac{1}{|x|} \cdot \Delta\phi(x)\, dx = -4\pi\phi(0).$$

The claim now follows.

Remark: We can interpret this calculation as giving us a rigorous justification of the formula that, on $\mathbb{R}^3$, one has:

$$\Delta\Big(\frac{1}{|x|}\Big) = -4\pi\delta_0$$

where δ_0 is the Dirac delta function. We would formally define $\Delta\Big(\frac{1}{|x|}\Big)$ to be the object which, when integrated against ϕ over $\mathbb{R}^3$ satisfies the following:

$$\int_{\mathbb{R}^3} \Delta\Big(\frac{1}{|x|}\Big) \cdot \phi(x)\, dx = \int_{\mathbb{R}^3} \frac{1}{|x|} \cdot \Delta\phi(x)\, dx.$$

(We are formally integrating by parts twice in the x variable.) From the earlier calculations, we know that this expression equals:

$$-4\pi\phi(0) = \int_{\mathbb{R}^3} \Big(-4\pi\delta_0(x)\Big) \cdot \phi(x)\, dx.$$

Hence:

$$\int_{\mathbb{R}^3} \Delta\Big(\frac{1}{|x|}\Big) \cdot \phi(x)\, dx = \int_{\mathbb{R}^3} \Big(-4\pi\delta_0(x)\Big) \cdot \phi(x)\, dx.$$

This holds for all functions ϕ which equal zero outside of some ball centered at the origin. Hence, we formally obtain:

$$\Delta\Big(\frac{1}{|x|}\Big) = -4\pi\delta_0.$$

b) If ϕ is assumed to be harmonic, then the integral on the right-hand side vanishes. Hence, we need to show that $\phi(0) = 0$.

Solution 1:

We can use the mean value property. Namely, we know that $\phi(0)$ equals the average of the function ϕ on $\partial B(0, 2R)$. However, the function ϕ vanishes on $\partial B(0, 2R)$, so $\phi(0) = 0$.

Solution 2:

We note that ϕ is a smooth function which vanishes outside of $B(0, R)$. It follows that ϕ is bounded. By Liouville's theorem, it follows that ϕ is constant. Since ϕ is equal to zero outside of $B(0, R)$, it follows that ϕ is equal to zero on all of $\mathbb{R}^3$. In particular $\phi(0) = 0$. $\Box$.

Exercise 3. *(Uniqueness of Green's functions)*

Suppose that $\Omega \subseteq \mathbb{R}^3$ is a bounded domain. Suppose that, for given $x_0 \in \Omega$, the functions $G^1(x, x_0)$ and $G^2(x, x_0)$, defined for $x \in \Omega \setminus \{x_0\}$, satisfy the conditions of the Green's function stated in class.

Prove that:

$$G^1(x, x_0) = G^2(x, x_0)$$

for all $x \in \Omega \setminus \{x_0\}$. In other words, the Green's function is uniquely defined.

Solution:

It is not possible to directly apply the uniqueness result for the Laplace's equation to the functions $G^1(x, x_0)$ and $G^2(x, x_0)$ since they are not harmonic on all of Ω. We can, however, modify this approach to prove the claim. Let us consider the functions:

$$u^1(x) := G^1(x, x_0) + \frac{1}{4\pi|x - x_0|}$$

and

$$u^2(x) := G^2(x, x_0) + \frac{1}{4\pi|x - x_0|}.$$

By construction, both u^1 and u^2 are harmonic on Ω.
Moreover, for $x \in \partial\Omega$, we know that:

$$G^1(x, x_0) = G^2(x, x_0) = 0$$

and hence:

$$u^1(x) = u^2(x) = \frac{1}{4\pi|x - x_0|}.$$

We can now apply the uniqueness result for Laplace's equation to the functions u^1 and u^2 in order to deduce that $u^1 = u^2$. From this equality, it follows that:

$$G^1(x, x_0) = G^2(x, x_0)$$

for all $x \in \Omega$. $\square$

Exercise 4. *(Equipartition of energy for the wave equation) Suppose that $g, h : \mathbb{R} \to \mathbb{R}$ are smooth functions which vanish outside of some interval of finite length and let $u \in C^2(\mathbb{R} \times [0, +\infty))$ solve the initial value problem for the wave equation in one dimension:*

$$\begin{cases} u_{tt} - u_{xx} = 0 \text{ on } \mathbb{R} \times (0, +\infty) \\ u = g, u_t = h \text{ on } \mathbb{R} \times \{t = 0\}. \end{cases} \tag{1}$$

Note that, in this case the constant c is assumed to equal 1.

*The **kinetic energy** of the solution u is defined by:*

$$k(t) := \frac{1}{2} \int_{-\infty}^{+\infty} u_t^2(x, t) dx$$

*and the **potential energy** of u is defined by:*

$$p(t) := \frac{1}{2} \int_{-\infty}^{+\infty} u_x^2(x, t) dx.$$

a) Show that $k(t) + p(t)$ is constant in time by using the formula from class:

$$u(x, t) = f(x - t) + g(x + t).$$

Hence, the total energy is conserved in time. We recall that, in class, we proved this fact directly by using the equation.

b) Moreover, show that $k(t) = p(t)$ for sufficiently large t. In other words, the total energy gets equally partitioned into the kinetic and potential part over a sufficiently long time.

Solution:

a) For $u(x, t) = f(x - t) + g(x + t)$, we compute:

$$u_t(x, t) = -f'(x - t) + g'(x + t)$$

and

$$u_x(x, t) = f'(x - t) + g'(x + t).$$

Let us denote the total energy by $E(t)$. Then, we obtain that:

$$E(t) = k(t) + p(t) = \frac{1}{2} \int_{-\infty}^{+\infty} \Big(- f'(x - t) + g'(x + t)\Big)^2 dx + \frac{1}{2} \int_{-\infty}^{+\infty} \Big(f'(x - t) + g'(x + t)\Big)^2 dx =$$

$$= \int_{-\infty}^{+\infty} \Big((f'(x-t))^2 + (g'(x+t))^2 - f'(x-t) \cdot g'(x+t) + f'(x-t) \cdot g'(x+t) \Big)\, dx =$$

$$= \int_{-\infty}^{+\infty} \Big(f'(x-t) \Big)^2 dx + \int_{-\infty}^{+\infty} \Big(g'(x+t) \Big)^2 dx =$$

$$= \int_{-\infty}^{+\infty} \Big(f'(x) \Big)^2 dx + \int_{-\infty}^{+\infty} \Big(g'(x) \Big)^2 dx = E(0).$$

Hence, the energy is conserved in time.

b) We calculate as before:

$$k(t) = \frac{1}{2} \int_{-\infty}^{+\infty} \Big((f'(x-t))^2 + (g'(x+t))^2 - 2f'(x-t) \cdot g'(x+t) \Big)\, dx.$$

and:

$$p(t) = \frac{1}{2} \int_{-\infty}^{+\infty} \Big((f'(x-t))^2 + (g'(x+t))^2 + 2f'(x-t) \cdot g'(x+t) \Big)\, dx.$$

We note that the integrands are the same when:

$$f'(x-t) \cdot g'(x+t) = 0.$$

We recall that the functions f and g equal zero outside of the interval $[-R, R]$ for some $R > 0$. In particular $f' = g' = 0$ outside of $[-R, R]$.

We note that $(x+t) - (x-t) = 2t$. Hence, if $t > R$, it is not possible for both $x - t$ and $x + t$ to be in $[-R, R]$. In particular, it follows that either $f'(x-t) = 0$ or $g'(x+t) = 0$, and so $f'(x-t) \cdot g'(x+t) = 0$ for all $x \in \mathbb{R}$, whenever $t > R$.

Hence, we may conclude that:

$$k(t) = p(t)$$

for all $t > R$, where R is defined as above. □