# MATH 425, HOMEWORK 6, SOLUTIONS

**Exercise 1.** *(Uniqueness for the Poisson equation by using the energy method)*
*Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain. We assume that for all $f : \Omega \to \mathbb{R}$ and for all $g : \partial\Omega \to \mathbb{R}$, the boundary value problem:*

$$\begin{cases} \Delta u = f \text{ on } \Omega \\ u = g \text{ on } \partial\Omega \end{cases}$$

*admits a solution.*
*By using the energy method, show that this solution is uniquely determined if we are given $f$ and $g$.*

**Solution:**
We suppose that $u_1$ and $u_2$ solve:

$$\begin{cases} \Delta u_1 = f \text{ on } \Omega \\ u_1 = g \text{ on } \partial\Omega \end{cases}$$

and

$$\begin{cases} \Delta u_2 = f \text{ on } \Omega \\ u_2 = g \text{ on } \partial\Omega \end{cases}$$

Let $w := u_1 - u_2$. Then, $w$ solves:

$$\begin{cases} \Delta w = 0 \text{ on } \Omega \\ w = 0 \text{ on } \partial\Omega \end{cases}$$

We want to argue that $w = 0$ because then it follows that $u_1 = u_2$.
Since $\Delta w = 0$, it follows that:

$$\int_\Omega \Delta w \cdot w \, dx\, dy\, dz = 0$$

We can use *Green's First Identity* from multivariable calculus to deduce that:

$$\int_\Omega -|\nabla w|^2 \, dx\, dy\, dz + \int_{\partial\Omega} w \nabla w \cdot \vec{n} \, dS = \int_\Omega \Delta w \cdot w \, dx\, dy\, dz = 0.$$

Here, $\vec{n}$ is the outward pointing unit normal vector to $\partial\Omega$. Since $w = 0$ on $\partial\Omega$, it follows that:

$$\int_\Omega -|\nabla w|^2 \, dx\, dy\, dz = 0.$$

In particular, we obtain:

$$\nabla w = 0 \text{ on } \Omega.$$

Hence, $w$ is constant (on each connected component) of $\Omega$. Since $w = 0$ on $\partial\Omega$, it follows that $w = 0$, as was claimed. $\square$

**Exercise 2.** *(A necessary condition for existence of solutions)*
*Suppose that $\Omega \subseteq \mathbb{R}^3$ is a bounded domain and suppose that $f : \Omega \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$. Consider the boundary value problem:*

$$\begin{cases} \Delta u = f \text{ on } \Omega \\ \frac{\partial u}{\partial n} = g \text{ on } \partial\Omega. \end{cases}$$

*Show that the above boundary value problem doesn't have a solution unless:*

$$\int_{\Omega} f\,dx\,dy\,dz = \int_{\partial\Omega} g\,dS$$

*Here, we recall that* $\frac{\partial u}{\partial n} := \nabla u \cdot \vec{n}$, *where* $\vec{n}$ *is the outward pointing unit normal vector to* $\partial\Omega$.

**Solution:** Suppose that $u$ solves the given boundary value problem. We then observe that:

$$\int_{\Omega} f\,dx\,dy\,dz = \int_{\Omega} \Delta u\,dx\,dy\,dz.$$

Let us note that $\Delta u = \div \nabla u$. Hence, by the Divergence Theorem, it follows that:

$$\int_{\Omega} \Delta u\,dx\,dy\,dz = \int_{\partial\Omega} \nabla u \cdot \vec{n}\,dS$$

We know that:

$$\nabla \cdot \vec{n} = \frac{\partial u}{\partial n} = g.$$

Hence:

$$\int_{\partial\Omega} \nabla u \cdot \vec{n}\,dS = \int_{\partial\Omega} g\,dS.$$

Combining the above equalities, it follows that:

$$\int_{\Omega} f\,dx\,dy\,dz = \int_{\partial\Omega} g\,dS. \square$$

**Exercise 3.** *(Subharmonic functions)*
*We say that a function* $u = u(x)$ *is subharmonic if* $\Delta u \geq 0$. *In particular, every harmonic function is subharmonic. In this exercise, we will study the maximum principle for subharmonic functions.*

*a) Suppose that* $\Omega \subseteq \mathbb{R}^n$ *is a bounded domain and suppose that* $u$ *is a subharmonic function on* $\Omega$. *Furthermore, assume that* $u$ *extends to a continuous function on* $\bar{\Omega} = \Omega \cup \partial\Omega$.
*Show that* $u$ *achieves its maximum value on* $\partial\Omega$. *In other words:*

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u.$$

*b) Fix* $n = 2$ *and look at the function* $u(x_1, x_2) = x_1^2 + x_2^2$ *on the closed unit ball*

$$B(0,1) = \{(x_1, x_2) \in \mathbb{R}^2,\ x_1^2 + x_2^2 \leq 1\}.$$

*Calculate* $\Delta u$ *and deduce that* $u$ *is subharmonic.*

*c) Check that the maximum principle holds for the function* $u$ *defined in part b) when the domain* $\Omega$ *is the open unit ball:* $\{(x_1, x_2) \in \mathbb{R}^2,\ x_1^2 + x_2^2 < 1\}$.

*d) For the function* $u$ *defined in part b), find where it achieves its minimum on* $B(0,1)$. *Is this minimum achieved on the boundary?*

**Solution:**

a) If $u$ were to achieve its maximum at an interior point $x_0 \in \Omega$, we would know that $u_{x_j x_j}(x_0) \leq 0$ for all $j = 1, 2, \ldots, n$. In particular, we would obtain that $\Delta u(x_0) \leq 0$. This doesn't immediately lead to a contradiction since $\Delta u(x_0)$ can equal zero.

We modify this argument by considering a good approximation of $u$. Given $\epsilon > 0$, we define the function $v^\epsilon : \Omega \to \mathbb{R}$ by:

$$v^\epsilon(x) := u(x) + \epsilon |x|^2.$$

We note that $v^\epsilon$ extends to a continuous function on $\overline{\Omega} = \Omega \cup \partial\Omega$ for all $\epsilon > 0$.

We observe that:

$$\Delta v^{\epsilon} = \Delta u + \Delta(\epsilon|x|^2) = \Delta u + 2\epsilon n \geq 2\epsilon n > 0.$$

In particular, the above argument tells us that $v^{\epsilon}$ must achieve its maximum on $\overline{\Omega}$ on $\partial\Omega$. We note that the maximum exists since $\Omega$ is assumed to be bounded and since $v^{\epsilon}$ extends to a continuous function on $\overline{\Omega}$. Given $\epsilon > 0$, let us denote by $x^{\epsilon}$ the element of $\partial\Omega$ at which $v^{\epsilon}$ achieves its maximum. Moreover, let us denote by $x_M$ the point such that:

$$u(x_M) = \max_{y \in \partial\Omega} u(y).$$

In other words, $x_M$ is the point on $\partial\Omega$ on which $u$ achieves its maximum. As before, we note that this maximum is achieved.

Let us fix a point $x \in \Omega$. We want to show that:

$$u(x) \leq u(x_M). \tag{1}$$

We know that, for all $\epsilon > 0$:

$$v^{\epsilon}(x) \leq v^{\epsilon}(x_{\epsilon}). \tag{2}$$

Let us note that, by construction of $v^{\epsilon}$ and (2), it follows that:

$$u(x) = v^{\epsilon}(x) - \epsilon|x|^2 \leq v^{\epsilon}(x_{\epsilon}) - \epsilon|x|^2 = u(x_{\epsilon}) + \epsilon|x_{\epsilon}|^2 - \epsilon|x|^2.$$

Since $x_{\epsilon} \in \partial\Omega$, we know by construction of $x_M$ that:

$$u(x_{\epsilon}) \leq u(x_M).$$

Moreover, since $\Omega$ is bounded, we note that there exists $L > 0$ such that $|y| \leq L$ for all $y \in \overline{\Omega}$. Consequently:

$$u(x) \leq u(x_M) + \epsilon\,(L^2 - |x|^2). \tag{3}$$

We note that $L^2 - |x|^2 \geq 0$, so we can't immediately deduce the claim form (3). However, we can let $\epsilon \to 0$ in (3) and deduce that:

$$u(x) \leq u(x_M)$$

as was claimed.

b) We compute that:

$$\Delta u = \Delta(x_1^2 + x_2^2) = (x_1^2)_{x_1x_1} + (x_2^2)_{x_2x_2} = 2 + 2 = 4 > 0.$$

It follows that $u$ is subharmonic.

c) We note that, on a circle of radius $r$, the function $u$ equals $r^2$. In particular, if we look at $\overline{\Omega}$, the function $u$ has maximum 1 which is achieved on the circle of radius 1, which is the boundary of $\Omega$.

d) The function $u$ has a minimum at the origin, where it equals zero. The origin doesn't lie on the boundary of $\Omega$. $\square$