MATH 425, HOMEWORK 6

This homework is due by noon on Friday, March 22. Please leave your assignment in my mailbox. There are three problems. Each problem is worth 10 points.

Exercise 1. *(Uniqueness for the Poisson equation by using the energy method)*
Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain. We assume that for all $f : \Omega \to \mathbb{R}$ and for all $g : \partial\Omega \to \mathbb{R}$, the boundary value problem:

$$\begin{cases} \Delta u = f \ \text{ on } \Omega \\ u = g \ \text{ on } \partial\Omega \end{cases}$$

admits a solution.
By using the energy method, show that this solution is uniquely determined if we are given f and g.

[HINT: Suppose that u_1, u_2 are two solutions. Look at their difference $w := u_1 - u_2$ and find the problem which problem w solves. Multiply the equation for w by w and integrate over Ω. It is helpful to recall Green's Identities from multivariable calculus.]

Exercise 2. *(A necessary condition for existence of solutions)*
Suppose that $\Omega \subseteq \mathbb{R}^3$ is a bounded domain and suppose that $f : \Omega \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$. Consider the boundary value problem:

$$\begin{cases} \Delta u = f \ \text{ on } \Omega \\ \frac{\partial u}{\partial n} = g \ \text{ on } \partial\Omega. \end{cases}$$

Show that the above boundary value problem doesn't have a solution unless:

$$\int_\Omega f \, dx \, dy \, dz = \int_{\partial\Omega} g \, dS$$

Here, we recall that $\frac{\partial u}{\partial n} := \nabla u \cdot \vec{n}$, where $\vec{n}$ is the outward pointing unit normal vector to $\partial\Omega$. [HINT: Integrate the equation over Ω.]

Exercise 3. *(Subharmonic functions)*
We say that a function $u = u(x)$ is subharmonic if $\Delta u \geq 0$. In particular, every harmonic function is subharmonic. In this exercise, we will study the maximum principle for subharmonic functions.

a) Suppose that $\Omega \subseteq \mathbb{R}^n$ is a bounded domain and suppose that u is a subharmonic function on Ω. Furthermore, assume that u extends to a continuous function on $\bar{\Omega} = \Omega \cup \partial\Omega$.
Show that u achieves its maximum value on $\partial\Omega$. In other words:

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u.$$

b) Fix $n = 2$ and look at the function $u(x_1, x_2) = x_1^2 + x_2^2$ on the closed unit ball

$$B(0,1) = \{(x_1, x_2) \in \mathbb{R}^2, \ x_1^2 + x_2^2 \leq 1\}.$$

Calculate Δu and deduce that u is subharmonic.

c) Check that the maximum principle holds for the function u defined in part b) when the domain Ω is the open unit ball: $\{(x_1, x_2) \in \mathbb{R}^2, \ x_1^2 + x_2^2 < 1\}$.

d) For the function u defined in part b), find where it achieves its minimum on $B(0,1)$. Is this minimum achieved on the boundary?