MATH 425, HOMEWORK 5, SOLUTIONS

Exercise 1. *(Uniqueness for the heat equation on $\mathbb{R}$)*

Suppose that the functions $u_1, u_2 : \mathbb{R}_x \times \mathbb{R}_t \to \mathbb{R}$ solve:

$$\begin{cases} \partial_t u_1 - k \cdot \partial_x^2 u_1 = 0,\ x \in \mathbb{R},\, t > 0 \\ u_1(x,0) = \phi(x),\, x \in \mathbb{R} \end{cases}$$

and

$$\begin{cases} \partial_t u_2 - k \cdot \partial_x^2 u_2 = 0,\ x \in \mathbb{R},\, t > 0 \\ u_2(x,0) = \phi(x),\, x \in \mathbb{R} \end{cases}$$

for some function $\phi : \mathbb{R} \to \mathbb{R}$.
Suppose furthermore that there exists constants $C, A > 0$ such that for all $x \in \mathbb{R}$ and for all $t > 0$, one has:

$$|u_1(x,t)| \leq Ce^{Ax^2} \text{ and } |u_2(x,t)| \leq Ce^{Ax^2}.$$

Using the Global Maximum Principle (which was stated in class), show that:

$$u_1 = u_2.$$

(Here, one is allowed to use the result of the Global Maximum Principle, even though we didn't give the details of its proof in class.)

This type of result is called ***Conditional Uniqueness****. In other words, we know that solutions are unique in the class of objects satisfying some additional condition, which in this case is a bound of the type $|u(x,t)| \leq Ce^{Ax^2}$.*

Solution: Let $w(x,t) := u_1(x,t) - u_2(x,t)$. We want to show that $w(x,t) = 0$ for all $x \in \mathbb{R}$ and for all $t > 0$.

Let us note that w solves:

$$\begin{cases} \partial_t w - k \cdot \partial_x^2 w = 0,\ x \in \mathbb{R},\, t > 0 \\ w(x,0) = 0,\, x \in \mathbb{R}. \end{cases} \tag{1}$$

Moreover, by the triangle inequality, we note that, for all $x \in \mathbb{R}$ and for all $t > 0$:

$$|w(x,t)| = |u_1(x,t) - u_2(x,t)| \leq |u_1(x,t)| + |u_2(x,t)| \leq 2Ce^{Ax^2}$$

by using the assumptions on u_1 and u_2. In particular, it follows that w belongs to the class of functions for which we can apply the global maximum principle (the fact that the constant C gets replaced by $2C$ doesn't matter here). Hence, we can apply the global maximum principle to (2) in order to deduce that w achieves its maximum and minimum for $t = 0$. Since $w(x,0) = 0$ for all $x \in \mathbb{R}$, it follows that w is identically equal to zero. The claim now follows. $\square$.

Exercise 2. *(The Global Maximum Principle in a special case)*

In this Exercise, we will give a proof of a special case of the Global Maximum Principle.
Suppose that

$$\begin{cases} u_t - ku_{xx} = 0,\ x \in \mathbb{R},\, t > 0 \\ u(x,0) = \phi(x),\, x \in \mathbb{R} \end{cases}$$

for some bounded continuous function $\phi : \mathbb{R} \to \mathbb{R}$ *which equals zero outside of* $[0, 1]$.
Suppose moreover that the solution u *is bounded from above, i.e. that there exists* $C > 0$ *such that:*

$$u(x,t) \leq C \text{ for all } x \in \mathbb{R},\ t > 0.$$

Let M_0 *denote the maximum of* ϕ *(which exists by the assumptions on* ϕ*).*

We want to prove that:

$$u(x,t) \leq M_0 \text{ for all } x \in \mathbb{R},\ t > 0. \tag{2}$$

a) Fix $T > 0, L > 0$ *and consider the rectangle:* $Q_{L,T} := [-L, L]_x \times [0, T]_t$. *Define on* $Q_{T,L}$ *the function:*

$$w(x,t) := \frac{2C}{L^2} \cdot \Big(\frac{x^2}{2} + kt\Big) + M_0$$

Check that:

$$w_t - kw_{xx} = 0.$$

b) Explain how we can deduce that:

$$w \geq u \text{ on } Q_{L,T}.$$

[HINT: Recall the comparison results from the previous homework assignment.]

c) Fix (x_0, t_0). *By using the result from part b), and by letting* L *tend to infinity, deduce the bound* (2).

Solution:

a) We compute:

$$w_t(x,t) = \frac{2Ck}{L^2}$$

and

$$w_{xx}(x,t) = \frac{2C}{L^2}.$$

It immediately follows that $w_t - kw_{xx} = 0$.

b) By using Exercise 3a) from Homework assignment 4, it suffices to check that $w \geq u$ for $t = 0$ and for $x = \pm L$. More precisely, we look at the point:

- $(x, 0)$ for $-L \leq x \leq L$. Then:

  $$w(x,0) = \frac{Cx^2}{L^2} + M_0 \geq M_0 \geq \phi(x) = u(x,0).$$

  Here, we used the assumption that M_0 was the maximum of ϕ.
- (L, t) for $0 \leq t \leq T$. Here:

  $$w(L,t) = C + \frac{2Ckt}{L^2} + M_0 \geq C \geq u(L,t).$$

  Here, we used the assumption that u was bounded from above by C.
- $(-L, t)$ for $0 \leq t \leq T$. Since w is even in the x variable, this case is analogous to the previous one.

It follows that:

$$w \geq u \text{ on } Q_{L,T}.$$

c) We fix $(x_0, t_0) \in \mathbb{R} \times (0, +\infty)$. Let us find $L, T > 0$ sufficiently large such that $(x_0, t_0) \in Q_{L,T}$. From part b), it follows that:

$$u(x_0, t_0) \leq \frac{2C}{L^2} \cdot \Big(\frac{x_0^2}{2} + kt_0\Big) + M_0.$$

We can now let $L \to \infty$ in the above inequality and we deduce that:

$$u(x_0, t_0) \leq M_0.$$

(We note that we didn't need to let $T \to \infty$ in this step.) Since (x_0, t_0) is arbitrary, the claim follows. $\square$

Exercise 3. *(Separation of variables for an inhomogeneous PDE)*
a) Solve the following boundary value problem by using the method of separation of variables:

$$\begin{cases} u_t - u_{xx} = \sin(2\pi x) + \sin(3\pi x), \ \ 0 < x < 1, \, t > 0 \\ u(x,0) = 0, \, 0 \leq x \leq 1 \\ u(0,t) = u(1,t) = 0, \, t > 0. \end{cases}$$

b) Solve the more general problem:

$$\begin{cases} v_t - v_{xx} = m \cdot v + \sin(2\pi x) + \sin(3\pi x), \ \ 0 < x < 1, \, t > 0 \\ v(x,0) = 0, \, 0 \leq x \leq 1 \\ v(0,t) = v(1,t) = 0, \, t > 0 \end{cases}$$

for $m \in \mathbb{R}$ a constant.

Solution:

Let us note that, by uniqueness, there is only one solution to each given boundary value problem. Hence, in both parts a) and b), we need to construct a solution and this solution will then be unique.

a) We look for a solution of the form:

$$u(x,t) = A(t) \cdot \sin(2\pi x) + B(t) \cdot \sin(3\pi x). \tag{3}$$

The reason why we look for such a solution is that the right-hand side of the equation contains the $\sin(2\pi x)$ and $\sin(3\pi x)$ terms. We expect that these are the only frequencies that will be present in the solution. In the form of u that we are looking for, for each fixed t, the function $u(x,t)$ has a Fourier sine expansion in terms of $\sin(2\pi x)$ and $\sin(3\pi x)$. The coefficients will be functions of t.

Let us note that, for u defined as in (3), the boundary conditions $u(0,t) = u(1,t) = 0$ are satisfied since $\sin(0) = \sin(2\pi) = \sin(3\pi) = 0$.

Our goal is to choose $A(t)$ and $B(t)$ such that u solves the inhomogeneous heat equation. We compute:

$$u_t - u_{xx} = \Big\{A'(t) + 4\pi^2 A(t)\Big\} \cdot \sin(2\pi x) + \Big\{B'(t) + 9\pi^2 B(t)\Big\} \cdot \sin(3\pi x)$$

which, by the equation, equals:

$$\sin(2\pi x) + \sin(3\pi x).$$

We can now equate coefficients of $\sin(2\pi x)$ and $\sin(3\pi x)$ to deduce:

$$\begin{cases} A'(t) + 4\pi^2 A(t) = 1 \\ B'(t) + 9\pi^2 B(t) = 1. \end{cases} \tag{4}$$

Hence, the condition (4) guarantees that the function u defined in (3) solves the PDE.

We now need to solve for $A(t)$ and $B(t)$.

By the condition that $u(x,0) = A(0) \cdot \sin(2\pi x) + B(0) \cdot \sin(3\pi x)$, it follows that $A(0) = B(0)$.

Hence, we need to solve the following initial value problem to determine $A(t)$:

$$\begin{cases} A'(t) + 4\pi^2 A(t) = 1 \\ A(0) = 0. \end{cases}$$

We solve the ODE by multiplying with the integrating factor $e^{4\pi^2 t}$. The ODE then becomes:

$$e^{4\pi^2 t} A'(t) + 4\pi^2 e^{4\pi^2 t} A(t) = e^{4\pi^2 t}$$

i.e.

$$(e^{4\pi^2 t} A(t))' = e^{4\pi^2 t}.$$

Hence:

$$e^{4\pi^2 t} A(t) = A_0 + \frac{1}{4\pi^2} e^{4\pi^2 t}.$$

We note that $A(0) = 0$ implies that $A_0 = -\frac{1}{4\pi^2}$. Consequently:

$$A(t) = \frac{1}{4\pi^2} \cdot \Big\{1 - e^{-4\pi^2 t}\Big\}.$$

Similarly, we obtain:

$$B(t) = \frac{1}{9\pi^2} \cdot \Big\{1 - e^{-9\pi^2 t}\Big\}.$$

It follows that:

$$u(x,t) = \frac{1}{4\pi^2} \cdot \Big\{1 - e^{-4\pi^2 t}\Big\} \cdot \sin(2\pi x) + \frac{1}{9\pi^2} \cdot \Big\{1 - e^{-9\pi^2 t}\Big\} \cdot \sin(3\pi x).$$

b) Let us look at the function $w(x,t) := e^{-mt} v(x,t)$. Then, we note that:

$$w_t - w_{xx} = e^{-mt} \cdot \Big\{v_t - v_{xx} - mv\Big\}.$$

Hence, if we multiply the equation for v and the initial and boundary conditions by e^{-mt} (for $t = 0$, we will just multiply everything by 1), it follows that w solves the problem:

$$\begin{cases} w_t - w_{xx} = e^{-mt} \cdot \sin(2\pi x) + e^{-mt} \cdot \sin(3\pi x),\ 0 < x < 1,\ t > 0 \\ w(x,0) = 0,\ 0 \leq x \leq 1 \\ w(0,t) = w(1,t) = 0,\ t > 0. \end{cases}$$

We again look for a solution of the same form as in part a) :

$$w(x,t) = A(t) \cdot \sin(2\pi x) + B(t) \cdot \sin(3\pi x).$$

Again, the fact that $w(0,t) = w(1,t) = 0$ follows by construction.

For w as defined above, we obtain:

$$w_t - w_{xx} = \Big\{A'(t) + 4\pi^2 A(t)\Big\} \cdot \sin(2\pi x) + \Big\{B'(t) + 9\pi^2 B(t)\Big\} \cdot \sin(3\pi x) = e^{-mt} \cdot \sin(2\pi x) + e^{-mt} \cdot \sin(3\pi x)$$

It follows that:

$$\begin{cases} A'(t) + 4\pi^2 A(t) = e^{-mt} \\ B'(t) + 9\pi^2 B(t) = e^{-mt}. \end{cases}$$

As before, the condition $w(x,0) = 0$ implies that we need to take $A(0) = B(0) = 0$.

Hence, the initial value problem for $A(t)$ becomes:

$$\begin{cases} A'(t) + 4\pi^2 A(t) = e^{-mt} \\ A(0) = 0. \end{cases}$$

As in part a), we use the integrating factor $e^{4\pi^2 t}$ and deduce that:

$$(e^{4\pi^2 t} A(t))'(t) = e^{4\pi^2 t - mt}.$$

Let us first assume that $m \neq 4\pi^2$. In that case, we obtain:

$$e^{4\pi^2 t} A(t) = A_0 + \frac{1}{4\pi^2 - m} e^{4\pi^2 t - mt}.$$

Since $A(0) = 0$, it follows that $A_0 = -\frac{1}{4\pi^2 - m}$. Consequently:

$$A(t) = \frac{1}{4\pi^2 - m} \cdot \Big\{ e^{-mt} - e^{-4\pi^2 t} \Big\}.$$

If $m = 4\pi^2$, the ODE for $A(t)$ becomes:

$$(e^{4\pi^2 t} A(t))' = 1$$

Hence:

$$e^{4\pi^2 t} A(t) = A_0 + t.$$

It follows that $A_0 = 0$ and so:

$$A(t) = t e^{-4\pi^2 t}.$$

Similarly, if $m \neq 9\pi^2$, we obtain that:

$$B(t) = \frac{1}{9\pi^2 - m} \cdot \Big\{ e^{-mt} - e^{-9\pi^2 t} \Big\}.$$

If $m = 9\pi^2$, then:

$$B(t) = t e^{-9\pi^2 t}.$$

We put everything together to deduce that:

$$v(x,t) = e^{mt} \cdot w(x,t) = e^{mt} \cdot A(t) \cdot \sin(2\pi x) + e^{mt} \cdot B(t) \cdot \sin(3\pi x).$$

We need to consider three cases:

- $\mathbf{m \neq 4\pi^2}$, $\mathbf{m \neq 9\pi^2}$:

$$v(x,t) = e^{mt} \cdot \frac{1}{4\pi^2 - m} \cdot \Big\{ e^{-mt} - e^{-4\pi^2 t} \Big\} \cdot \sin(2\pi x) + e^{mt} \cdot \frac{1}{9\pi^2 - m} \cdot \Big\{ e^{-mt} - e^{-9\pi^2 t} \Big\} \cdot \sin(3\pi x).$$

- $\mathbf{m = 4\pi^2}$:

$$v(x,t) = e^{mt} \cdot t e^{-4\pi^2 t} \cdot \sin(2\pi x) + e^{mt} \cdot \frac{1}{9\pi^2 - m} \cdot \Big\{ e^{-mt} - e^{-9\pi^2 t} \Big\} \cdot \sin(3\pi x).$$

- $\mathbf{m = 9\pi^2}$:

$$v(x,t) = e^{mt} \cdot \frac{1}{4\pi^2 - m} \cdot \Big\{ e^{-mt} - e^{-4\pi^2 t} \Big\} \cdot \sin(2\pi x) + e^{mt} \cdot t e^{-9\pi^2 t} \cdot \sin(3\pi x).$$