MATH 425, HOMEWORK 4

This homework is due on Friday, February 22 (one should leave it in my mailbox by noon). There are three problems. Each problem is worth 10 points.

Exercise 1. *Consider the function $u(x,t) = 1 - x^2 - 2kt$, for $k > 0$ a constant.*

a) Verify that u is a solution to the heat equation.
b) Find the minimum and maximum of u on the closed rectangle $\{(x,t); 0 \leq x \leq 1,\, 0 \leq t \leq T\} = [0,1]_x \times [0,T]_t$ for a fixed $T > 0$, without using the maximum principle.
c) Find the minimum and maximum of u on $[0,1]_x \times [0,T]_t$ by using the maximum principle.

Exercise 2. *Consider the initial value problem:*

$$\begin{cases} u_t - u_{xx} = 0, \ \textit{for}\ 0 < x < 1,\ t > 0 \\ u(x,0) = 4x(1-x), \ \textit{for}\ 0 \leq x \leq 1 \\ u(0,t) = 0,\ u(1,t) = 0, \ \textit{for}\ t > 0. \end{cases}$$

Show that:

$$u(x,t) = u(1-x,t),$$

for all $0 \leq x \leq 1, t \geq 0$.

[HINT: What boundary value problem does the function $v(x,t) := u(1-x,t)$ solve?]

Exercise 3. *(A comparison principle)*
a) Suppose that u and v both solve the heat equation on $[0,L]_x \times (0,+\infty)_t$. Furthermore, suppose that $u \leq v$ for $t = 0$, for $x = 0$, and for $x = L$. Show that:

$$u \leq v \ \textit{on}\ [0,L]_x \times [0,+\infty)_t.$$

[HINT: Be careful to use the maximum principle on bounded rectangles.]

b) More generally, consider functions u and v which solve $u_t - ku_{xx} = f(x,t)$ and $v_t - kv_{xx} = g(x,t)$ on $[0,L]_x \times (0,+\infty)_t$. We assume moreover that $f \leq g$ on $[0,L]_x \times (0,+\infty)_t$ and that $u \leq v$ for $t = 0$, for $x = 0$, as well as for $x = L$. Show that $u \leq v$ on $[0,L]_x \times [0,+\infty)_t$.

c) Suppose that the function v satisfies the inequality $v_t - v_{xx} \geq \sin x$ on $[0,\pi]_x \times (0,+\infty)$. Moreover, assume that: $v(0,t) \geq 0, v(\pi,t) \geq 0$ for all $t > 0$ and $v(x,0) \geq \sin x$ for all $0 \leq x \leq \pi$. Show that:

$$v(x,t) \geq (1 - e^{-t}) \sin x$$

on $[0,\pi]_x \times [0,+\infty)_t$.