MATH 425, HOMEWORK 2

This homework is due in class on Thursday, January 31. Each problem is worth 10 points.

Exercise 1. *(The role of the diffusion coefficient)*
In this exercise, we will see how to justify the fact that we can set the diffusion coefficient in the heat equation to equal 1*. Furthermore, we will see the importance of the sign of the diffusion coefficient.*

Suppose that we are looking at a function $u : \mathbb{R}^n_x \times \mathbb{R}^+_t \to \mathbb{R}$ *such that:*

$$\begin{cases} u_t - k \cdot \Delta u = 0 \\ u\big|_{t=0} = \Phi. \end{cases} \tag{1}$$

for some constant $k > 0$ *and for some function* $\Phi = \Phi(x)$*.*
a) Suppose that u *is a solution of* (1)*. By appropriately choosing the constants* a *and* b*, construct a function* $v : \mathbb{R}^n_x \times \mathbb{R}^+_t$ *of the form* $v(x,t) = u(ax, bt)$ *which solves the equation* $v_t - \Delta v = 0$ *on* $\mathbb{R}^n_x \times \mathbb{R}^+_t$*.*
b) Express $v\big|_{t=0}$ *you obtained this way in terms of* Φ*.*
c) Explain briefly why this type of transformation can't give us a solution of the equation $w_t + \Delta w = 0$ *on* $\mathbb{R}^n_x \times \mathbb{R}^+_t$*.*
d) Take $w(x,t) = u(x, -t)$*, for* u *a solution of* (1)*. On what set is* w *defined?*
e) Show that the function w *defined in part d) solves* $w_t + k \cdot \Delta w = 0$ *on its domain of definition.*

Exercise 2. *(The heat equation with convection)*
a) Suppose that we know that the solution to the general heat equation initial value problem on $\mathbb{R}$*:*

$$\begin{cases} u_t - k \cdot u_{xx} = 0 \text{ for, } x \in \mathbb{R}, \ t > 0 \\ u(x,0) = \phi(x) \end{cases}$$

is given by:

$$u(x,t) = \int_{-\infty}^{+\infty} S(x-y,t)\phi(y)dy \tag{2}$$

for an explicitly determined function $S(x,t)$*. (The fact that this is indeed the case will be shown in class on Tuesday, January 29.)*
Assuming the formula (2)*, solve the initial value problem for the heat equation with convection:*

$$\begin{cases} u_t - k \cdot u_{xx} + V \cdot u_x = 0, \text{ for } x \in \mathbb{R}, \ t > 0 \\ u(x,0) = \psi(x). \end{cases}$$

Here, we are assuming that V *is a real constant. The answer should be given in terms of an integral involving the function* S*.*

(HINT: It is a good idea to use a "moving frame", i.e. to look at the function $\tilde{u}(x,t) := u(x + Vt, t)$*. In the new coordinate system, the* x *coordinate "moves" with time at speed* V*.)*
b) Give a brief explanation of what sort of physical phenomenon can be modeled by the equation in part a). (It is not necessary to derive the equation here; just give a one sentence description).

Exercise 3. *(A consequence of invariance under dilations)*
Suppose that $Q : \mathbb{R}_x \times \mathbb{R}_t^+ \to \mathbb{R}$ *is a function. For* $a > 0$*, we define the new function* $Q^{(a)} : \mathbb{R}_x \times \mathbb{R}_t^+ \to \mathbb{R}$ *by:*

$$Q^{(a)}(x,t) := Q(\sqrt{a} \cdot x, a \cdot t).$$

Suppose that, for all $a > 0$*, and for all* $x \in \mathbb{R}, t > 0$*, one has:*

$$Q^{(a)}(x,t) = Q(x,t).$$

Show that, for all $C > 0$*, the function* Q *is constant along the segment of the parabola given by:*

$$x = C\sqrt{t};\ t > 0.$$

This exercise justifies our guess that $Q(x,t)$ *is a function of* $\frac{x}{\sqrt{t}}$*.*

Exercise 4. *(The Gaussian integral)*
In this exercise, we will summarize some important properties of a specific definite integral which we will need to use in order to study the heat equation on $\mathbb{R}$*.*
a) Show that:

$$\int_0^{+\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

(HINT: Look at the product $\int_0^{+\infty} e^{-x^2} dx \int_0^{+\infty} e^{-y^2} dy$ *and apply polar coordinates.)*
b) Deduce that: $\int_{-\infty}^0 e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$ *and hence:* $\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$*.*
c) Use a change of variables in order to conclude that: $\int_{-\infty}^{+\infty} S(x,t) dx = 1$ *for the function* $S(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{|x|^2}{4kt}}$*, defined for* $x \in \mathbb{R}$ *and* $t > 0$*. Here, we are assuming that* $k > 0$ *is a given constant.*