MATH 425, FINAL EXAM SOLUTIONS

Each exercise is worth 50 points.

Exercise 1. *a) The operator $\mathcal{L}_1$ is defined on smooth functions of (x, y) by:*

$$\mathcal{L}_1(u) := \arctan(xy) \cdot u_{xx} + \sin(x^2y^2) \cdot u_{yy}.$$

Is the operator $\mathcal{L}_1$ linear? Prove your answer.

b) Does the answer change if we replace the operator $\mathcal{L}_1$ by the operator $\mathcal{L}_2$, which is given by:

$$\mathcal{L}_2(u) := u_{xx} + e^u \ ?$$

c) Find the general solution of the PDE $u_x + x^2u_y = 0$ by using the method of characteristics. Check that your solution solves the PDE. You don't need to show that these are all of the solutions.

Solution:

a) Given smooth functions u, v and constants a, b, we compute:

$$\mathcal{L}_1(au + bv) = \arctan(xy) \cdot \big(au + bv\big)_{xx} + \sin(x^2y^2) \cdot \big(au + bv\big)_{yy} =$$

$$= a\Big(\arctan(xy) \cdot u_{xx} + \sin(x^2y^2) \cdot u_{yy}\Big) + b\Big(\arctan(xy) \cdot u_{xx} + \sin(x^2y^2) \cdot u_{yy}\Big) =$$

$$= a\mathcal{L}_1(u) + b\mathcal{L}_1(v).$$

Hence, $\mathcal{L}_1$ is linear.

b) We note that $\mathcal{L}_2(0) = 1 \neq 0$, which implies that the operator is not linear. Namely, for a linear operator T, we know that $T(0) = 0$ if we substitute $a = b = 0$ into the definition of linearity.

c) The characteristic ODE is given by:

$$\frac{dy}{dx} = x^2.$$

The general solution is given by:

$$y(x) = \frac{x^3}{3} + C.$$

Hence, by using the method of characteristics, the solution u is given by:

$$u(x, y) = f\Big(y - \frac{x^3}{3}\Big)$$

for some (differentiable) function $f : \mathbb{R} \to \mathbb{R}$.

For u defined as above, we note that:

$$u_x(x, y) = -x^2 f'\Big(y - \frac{x^3}{3}\Big)$$

and

$$u_y(x, y) = f'\Big(y - \frac{x^3}{3}\Big).$$

Hence:

$$u_x + x^2u_y = -x^2 f'\Big(y - \frac{x^3}{3}\Big) + x^2 f'\Big(y - \frac{x^3}{3}\Big) = 0. \ \square$$

Exercise 2. *In this exercise, we would like to find a solution to the following initial value problem:*

$$\begin{cases} u_t - u_{xx} = 0, \text{ for } x \in \mathbb{R}\,, t > 0 \\ u(x,0) = x^2, \text{ for } x \in \mathbb{R}. \end{cases} \tag{1}$$

a) Let $v := u_{xxx}$. What initial value problem does v solve?

b) Use this observation to deduce that we can take $v = 0$ to be a solution of the initial value problem obtained in part a).

c) What does this tell us about the form of u?

d) Use the latter expression to find a solution of (1). Check that the obtained function solves (1).

e) Alternatively, write the formula for a solution of (1) involving the heat kernel on $\mathbb{R}$. Write the heat kernel explicitly in terms of exponentials. Don't simplify the integral.

Solution:

a) By the differentiation property of the heat equation, we deduce that v also solves the heat equation. We note that $v_{xxx}(x,0) = 0$. Hence, v solves the initial value problem:

$$\begin{cases} v_t - v_{xx} = 0, \text{ for } x \in \mathbb{R}\,, t > 0 \\ v(x,0) = 0, \text{ for } x \in \mathbb{R}. \end{cases}$$

b) We note that the function $v = 0$ solves the initial value problem in part a).

c) From part b), we observe that we can look for a solution to (1) of the form:

$$u(x,t) = A(t) + B(t) \cdot x + C(t) \cdot x^2 \tag{2}$$

for some (differentiable) functions $A, B, C : \mathbb{R}_t^+ \to \mathbb{R}$ satisfying $A(0) = B(0) = 0, C(0) = 1$.

d) We note that, for u of the form (2), one has:

$$u_t - u_{xx} = (A'(t) - 2C(t)) + B'(t) \cdot x + C'(t) \cdot x^2$$

Hence, such a u solves the heat equation if and only if:

$$\begin{cases} A'(t) = 2C(t) \\ B'(t) = 0 \\ C'(t) = 0. \end{cases}$$

From the latter two conditions, it follows that B and C are constant. Since $B(0) = 0$ and $C(0) = 1$, we deduce that:

$$B(t) = 0 \text{ and } C(t) = 1.$$

We now use the first condition to deduce that:

$$A'(t) = 2C(t) = 2.$$

Since $A(0) = 0$, we conclude that $A(t) = 2t$. Putting all of this together, we obtain:

$$u(x,t) = 2t + x^2. \tag{3}$$

We readily check that the function u defined in (3) solves the initial value problem (1). Namely: $u_t = u_{xx} = 2$, hence $u_t - u_{xx} = 0$ and $u(x,0) = 0 + x^2 = x^2$.

e) We use the formula from class and recall that we are taking the diffusion coefficient to equal to 1, and hence u given by:

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{4t}} \cdot y^2 \, dy \tag{4}$$

solves (1). □

Exercise 3. *a) Show that the function $u : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ defined by $u(x) := \log |x|$ is harmonic on $\mathbb{R}^2 \setminus \{0\}$.*

In the following, suppose that $\phi : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function which equals zero outside of some ball centered at the origin.

b) Prove that:

$$\lim_{\epsilon \to 0} \int_{\partial B(0,\epsilon)} \Big[\log |x| \cdot \frac{\partial \phi}{\partial n} - \frac{\partial}{\partial n}\Big(\log |x|\Big) \cdot \phi(x) \Big] \, dS(x) \to 2\pi \phi(0).$$

*for n being the unit normal on $\partial B(0,\epsilon)$ pointing **towards the origin**.*

c) Use the result from part b) in order to prove:

$$\phi(0) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log |x| \cdot \Delta \phi(x) \, dx.$$

Solution:

a) We write $\log |x|$ as $\log \sqrt{x_1^2 + x_2^2}$.

By the Chain Rule, it follows that:

$$(\log |x|)_{x_1} = \frac{1}{\sqrt{x_1^2 + x_2^2}} \cdot \frac{2x_1}{2\sqrt{x_1^2 + x_2^2}} = \frac{x_1}{x_1^2 + x_2^2}.$$

$$(\log |x|)_{x_1 x_1} = \frac{1}{x_1^2 + x_2^2} - \frac{2x_1^2}{(x_1^2 + x_2^2)^2}.$$

By symmetry:

$$(\log |x|)_{x_2 x_2} = \frac{1}{x_1^2 + x_2^2} - \frac{2x_2^2}{(x_1^2 + x_2^2)^2}.$$

Summing the previous two identities, we obtain:

$$\Delta \log |x| = 0.$$

Alternatively, we can use the formula for Laplace's operator in polar coordinates:

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

in order to deduce that:

$$\Delta \log r = \Big(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\Big) \log r = -\frac{1}{r^2} + \frac{1}{r^2} = 0.$$

b)

$$\lim_{\epsilon \to 0} \int_{\partial B(0,\epsilon)} \Big[\log |x| \cdot \frac{\partial \phi}{\partial n} - \frac{\partial}{\partial n}\Big(\log |x|\Big) \cdot \phi(x) \Big] \, dS(x) \to 2\pi \phi(0).$$

Let us first observe that, there exists $M > 0$ independent of ϵ such that, when ϵ is sufficiently small, it is the case that:

$$\Big|\frac{\partial \phi}{\partial n}\Big| \leq M.$$

Consequently:

$$\Big|\int_{\partial B(0,\epsilon)} \log|x|\cdot\frac{\partial\phi}{\partial n}\,dS(x)\Big| = \big|\log(\epsilon)\big|\cdot\Big|\int_{\partial B(0,\epsilon)}\frac{\partial\phi}{\partial n}\,dS(x)\Big| \leq \big|\log(\epsilon)\big|\cdot\int_{\partial B(0,\epsilon)}\Big|\frac{\partial\phi}{\partial n}\Big|\,dS(x) \leq 2\pi M\epsilon\cdot|\log(\epsilon)|.$$

Let us now observe that:

$$\lim_{x\to 0+}(x\log x) = 0.$$

This fact follows from L'Hôpital's rule since:

$$\lim_{x\to 0+}\big(x\log x\big) = \lim_{x\to 0+}\frac{\log x}{\left(\frac{1}{x}\right)} = \lim_{x\to 0+}\frac{(\log x)'}{\left(\frac{1}{x}\right)'} = \lim_{x\to 0+}\frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0+}(-x) = 0.$$

Consequently, the integral of the first term goes to zero as $\epsilon\to 0$.

We now need to look at the integral of the second term, Let us note that:

$$\frac{\partial}{\partial n}\log|x| = \big(\nabla\log|x|\big)\cdot n.$$

From the calculations in part a), it follows that:

$$\nabla\log|x| = \Big((\log|x|)_{x_1},(\log|x|)_{x_2}\Big) = \Big(\frac{x_1}{x_1^2+x_2^2},\frac{x_2}{x_1^2+x_2^2}\Big) = \frac{x}{|x|^2}.$$

By definition, we obtain that on $\partial B(0,\epsilon)$, one has:

$$n = -\frac{x}{\epsilon}.$$

Hence:

$$\frac{\partial}{\partial n}\log|x| = -\frac{1}{\epsilon}\cdot\frac{x\cdot x}{|x|^2} = -\frac{1}{\epsilon}.$$

Alternatively, we can use polar coordinates and see that:

$$\frac{\partial}{\partial n}\log|x| = -\frac{\partial}{\partial r}\log r = -\frac{1}{r} = -\frac{1}{\epsilon}$$

on $\partial B(0,\epsilon)$. It follows that:

$$\int_{\partial B(0,\epsilon)}\Big[-\frac{\partial}{\partial n}\Big(\log|x|\Big)\cdot\phi(x)\Big]\,dS(x) = \frac{1}{\epsilon}\int_{\partial B(0,\epsilon)}\phi(x)\,dS(x)\to 2\pi\phi(0)\text{ as }\epsilon\to 0.$$

c) Let us assume that $\phi = 0$ outside of $B(0,R)\subseteq\mathbb{R}^2$ and let $\epsilon\in(0,R)$ be given. We let:

$$\Omega_\epsilon := B(0,2R)\setminus B(0,\epsilon).$$

From part a), we know that, on $\mathbb{R}^2\setminus\{0\}$:

$$\Delta\log|x| = 0.$$

We now apply Green's second identity, noting that ϕ and $\log|x|$ are both smooth on Ω_ϵ in order to deduce that:

$$\int_{\Omega_\epsilon}\Big[\log|x|\cdot\Delta\phi(x) - \Delta\log|x|\cdot\phi(x)\Big]\,dx = \int_{\partial\Omega_\epsilon}\Big[\log|x|\cdot\frac{\partial\phi}{\partial n} - \frac{\partial}{\partial n}\Big(\log|x|\Big)\cdot\phi(x)\Big]\,dS(x).$$

We note that $\partial\Omega_\epsilon$ consists of two parts: $\partial B(0,\epsilon)$ and $\partial B(0,2R)$. Since, by assumption, ϕ vanishes near $\partial B(0,2R)$, it follows that the contribution to the right-hand side from the outer boundary $\partial B(0,2R)$ equals to zero. Moreover, we know that $\Delta\log|x| = 0$ on Ω_ϵ. Hence, it follows that:

$$\int_{\Omega_\epsilon}\log|x|\cdot\Delta\phi(x)\,dx = \int_{\partial B(0,\epsilon)}\Big[\log|x|\cdot\frac{\partial\phi}{\partial n} - \frac{\partial}{\partial n}\Big(\log|x|\Big)\cdot\phi(x)\Big]\,dS(x).$$

We note that $\Delta\phi = 0$ for $|x|\geq 2R$ and we deduce that:

$$\int_{|x|\geq\epsilon}\log|x|\cdot\Delta\phi(x)\,dx = \int_{\partial B(0,\epsilon)}\Big[\log|x|\cdot\frac{\partial\phi}{\partial n} - \frac{\partial}{\partial n}\Big(\log|x|\Big)\cdot\phi(x)\Big]\,dS(x).$$

We now let $\epsilon \to 0$ and we use the result from part b) in order to deduce that:

$$\int_{\mathbb{R}^2} \log|x| \cdot \Delta\phi(x)\, dx = 2\pi\phi(0).$$

The claim now follows. □

Exercise 4. *Let us recall the representation formula for harmonic functions in three dimensions:*

For $\Omega \subseteq \mathbb{R}^3$ a bounded domain, u a harmonic function on Ω which extends continuously up to $\partial\Omega$, and $x_0 \in \Omega$, the following formula holds:

$$u(x_0) = \frac{1}{4\pi} \int_{\partial\Omega} \Big[-u(x) \frac{\partial}{\partial n}\Big(\frac{1}{|x - x_0|}\Big) + \frac{1}{|x - x_0|} \frac{\partial u}{\partial n} \Big]\, dS(x).$$

Here, n denotes the outward pointing unit normal on $\partial\Omega$.

*In this exercise, one is allowed to use the representation formula **without proof**.*

a) State the mean value property for harmonic functions in three-dimensions.

b) Use the representation formula in order to prove the mean value property in three dimensions.

c) State the definition of the Green's function $G(x, x_0)$ for the Laplace operator on a three-dimensional domain Ω with x_0 a point in Ω.

d) Use the representation formula and properties of the Green's function to show that the harmonic function u defined in the beginning of the problem satisfies:

$$u(x_0) = \int_{\partial\Omega} u(x) \cdot \frac{\partial G(x, x_0)}{\partial n}\, dS(x).$$

Solution:

a) Let $x_0 \in \mathbb{R}^3$ and $R > 0$ are given and suppose that $u : B(x_0, R) \to \mathbb{R}^3 \to \mathbb{R}$ is a harmonic function which extends continuously up to $\partial B(x_0, R)$. The mean value property then states that:

$$u(x_0) = \frac{1}{4\pi R^2} \int_{\partial B(x_0,R)} u(y)\, dS(y).$$

b) We can replace the function u with the function $v(x) := u(x - x_0$ to see that it suffices to prove the claim in the special case when $x = 0$. It is important to note that the function v is harmonic if the function u is harmonic.

In other words, we are assuming that $u : B(0, R) \to \mathbb{R}$ is harmonic and that it extends continuously up to $\partial B(0, R)$ and we want to prove that:

$$u(0) = \frac{1}{4\pi R^2} \int_{\partial B(0,R)} u(y)\, dS(y)$$

by using the fact that:

$$u(0) = \frac{1}{4\pi} \int_{\partial B(0,R)} \Big[-u(y) \frac{\partial}{\partial n}\Big(\frac{1}{|y|}\Big) + \frac{1}{|y|} \frac{\partial u}{\partial n} \Big]\, dS(y).$$

Let us first note that we can use polar coordinates to deduce that, on $\partial B(0, R)$, we can write $\frac{\partial}{\partial n} = \frac{\partial}{\partial r}$. Hence:

$$\frac{\partial}{\partial n}\Big(\frac{1}{|y|}\Big) = \frac{\partial}{\partial r}\Big(\frac{1}{r}\Big) = -\frac{1}{r^2}.$$

It follows that:

$$\frac{1}{4\pi}\int_{\partial B(0,R)}\Big[-u(y)\frac{\partial}{\partial n}\Big(\frac{1}{|y|}\Big)\Big]\,dS(y)=\frac{1}{4\pi R^2}\int_{\partial B(0,R)}u(y)\,dS(y). \tag{5}$$

Moreover, we note that:

$$\frac{1}{4\pi}\int_{\partial B(0,R)}\frac{1}{|y|}\frac{\partial u}{\partial n}\,dS(y)=\frac{1}{4\pi R}\int_{\partial B(0,R)}\nabla u\cdot n\,dS(y)$$

which, by the Divergence Theorem equals:

$$\frac{1}{4\pi R}\int_{B(0,R)}\Delta u(y)\,dy=0. \tag{6}$$

The claim now follows from (5) and (6).

c) Suppose that $\Omega\subseteq\mathbb{R}^3$ is a bounded domain with $x_0\in\Omega$. The Green's function $G(x,x_0)$ is a function defined on $\Omega\setminus\{x_0\}$, which is continuous up to $\partial\Omega$, and which satisfies the following properties:

1) $G(x,x_0)$ is a harmonic function on $\Omega\setminus\{x_0\}$.
2) $G(x,x_0)=0$ for $x\in\partial\Omega$.
3) $H(x,x_0):=G(x,x_0)+\frac{1}{4\pi|x-x_0|}$ is harmonic on Ω.

d) We recall from property 3) in part c) that the function $H(x,x_0)=G(x,x_0)+\frac{1}{4\pi|x-x_0|}$ is harmonic on Ω. We also know that the function u is harmonic on Ω. Hence, by Green's second identity:

$$0=\int_{\partial\Omega}\Big[u(x)\frac{\partial H(x,x_0)}{\partial n}-H(x,x_0)\frac{\partial u}{\partial n}\Big]\,dS(x).$$

By the representation formula, we know that:

$$u(x_0)=\int_{\partial\Omega}\Big[u(x)\frac{\partial}{\partial n}\Big(-\frac{1}{4\pi|x-x_0|}\Big)+\frac{1}{4\pi|x-x_0|}\frac{\partial u}{\partial n}\Big]\,dS(x).$$

By using property 3) of Green's functions, it follows that:

$$u(x_0)=\int_{\partial\Omega}\Big[u(x)\frac{\partial G(x,x_0)}{\partial n}-G(x,x_0)\frac{\partial u}{\partial n}\Big]\,dS(x)$$

By property 2), we know that $G(x,x_0)=0$ for $x\in\partial\Omega$. Hence:

$$u(x_0)=\int_{\partial\Omega}u(x)\frac{\partial G(x,x_0)}{\partial n}\,dS(x)$$

as was claimed. □

Exercise 5. *Throughout this exercise, we assume that $c>0$ is a constant.*

a) Consider the differential operator $\frac{\partial^2}{\partial t^2}-c^2\frac{\partial^2}{\partial x^2}$, defined on smooth functions of $(x,t)\in\mathbb{R}\times\mathbb{R}$.

Show that there exist first-order differential operators T_1 and T_2 such that for all smooth functions $u:\mathbb{R}_x\times\mathbb{R}_t\to\mathbb{R}$, the following identity holds:

$$\Big(\frac{\partial^2}{\partial t^2}-c^2\frac{\partial^2}{\partial x^2}\Big)u=T_1T_2u.$$

b) What is the physical interpretation of the operators T_1 and T_2?

c) Using the above factorization, show that the general solution to the wave equation on $\mathbb{R}_x\times\mathbb{R}_t$:

$$u_{tt}-c^2u_{xx}=0$$

is given by:

$$u(x,t)=f(x-ct)+g(x+ct)$$

for some functions $f, g : \mathbb{R} \to \mathbb{R}$.

d) Check that the function u obtained in part c) solves the wave equation. How many derivatives do the functions f and g need to have in order for this calculation to be rigorous?

Solution:

a) We note that:

$$\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} = \Big(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\Big)\Big(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\Big).$$

More precisely, for $u : \mathbb{R}_x \times \mathbb{R}_t \to \mathbb{R}$, the following identity holds:

$$\Big(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\Big)\Big(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\Big)u = \Big(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\Big)\Big(u_t + cu_x\Big) =$$

$$= u_{tt} + cu_{xt} - cu_{tx} - c^2 u_{xx} = u_{tt} - c^2 u_{xx} = \Big(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\Big)u.$$

Hence, we can take:

$$T_1 = \frac{\partial}{\partial t} - c\frac{\partial}{\partial x}$$

$$T_2 = \frac{\partial}{\partial t} + c\frac{\partial}{\partial x}.$$

b) The operators T_1 and T_2 correspond to transport with speed c to the left and to the right respectively.

c) Suppose that:

$$u_{tt} - c^2 u_{xx} = 0.$$

By part a), we can write this equation as:

$$\Big(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\Big)\Big(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\Big)u = 0.$$

Let us take:

$$G(x,t) := u_t + cu_x.$$

We can then deduce that:

$$G_t - cG_x = 0.$$

Hence, G solves the transport equation. It follows that:

$$G(x,t) = H(x+ct)$$

for some (differentiable) function $H : \mathbb{R} \to \mathbb{R}$. We substitute this back into the definition of the function G to deduce that u then has to solve:

$$u_t + cu_x = G(x,t) = H(x+ct). \tag{7}$$

In particular, u solves an inhomogeneous transport equation. We note that the general solution of the associated homogeneous equation:

$$u_t^{(h)} + cu_x^{(h)} = 0$$

is given by:

$$u^{(h)}(x,t) = f(x-ct)$$

for some (differentiable) function $f : \mathbb{R} \to \mathbb{R}$. Hence, we need to find a particular solution $u^{(p)}$ of (7). Since the right-hand side is a function of $x+ct$, we look for a particular solution which is a function of $x+ct$ as well. In particular, we look for a solution of the form:

$$u^{(p)}(x,t) = g(x+ct)$$

for some (differentiable) function $g : \mathbb{R} \to \mathbb{R}$. For $u^{(p)}$ defined as above, we note that:

$$u_t^{(p)} + cu_x^{(p)} = (1+c) \cdot g'(x+ct).$$

Hence, we want to choose h in such a way that:

$$(1+c)g'(x+ct) = H(x+ct).$$

In particular, we can take:

$$g(y) := \frac{1}{1+c} \int_0^y H(s)\, ds.$$

Consequently, we obtain that:

$$u(x,t) = f(x-ct) + g(x+ct).$$

d) For u defined as in part c), we note that:

$$\begin{aligned} u_t &= -cf'(x-ct) + cg'(x+ct) \\ u_{tt} &= c^2 f''(x-ct) + c^2 g''(x+ct) \\ u_x &= f'(x-ct) + g'(x+ct) \\ u_{xx} &= f''(x-ct) + g''(x-ct). \end{aligned}$$

In particular, it follows that:

$$u_{tt} = c^2 u_{xx} = c^2 f''(x-ct) + c^2 g''(x+ct)$$

and so:

$$u_{tt} - c^2 u_{xx} = 0.$$

In order to make this calculation rigorous, we need to assume that the functions $f, g : \mathbb{R} \to \mathbb{R}$ are twice differentiable. $\square$