CLASS OF 1880 EXAM

Friday, April 5, 2013.
The duration of this test is two hours. No books, notes or calculators are allowed.
Each problem is worth 25 points. It is necessary to show all of your work for full credit.

Exercise 1. *Suppose that* $\square ABCD$ *is a trapezoid whose sides* AB *and* CD *are parallel. Let* S *denote the intersection of the diagonals of* $\square ABCD$*. We denote by* A_1, A_2, A_3, A_4 *the areas of the triangles* $\Delta ABS, \Delta BCS, \Delta CDS, \Delta DAS$ *respectively.*

a) Prove that: $A_2 = A_4$.
b) Moreover, prove that: $A_1 \cdot A_3 = A_2^2$
c) Let A *denote the area of* $\square ABCD$*. Prove that:* $A \geq 4A_2$.
d) What can one say about $\square ABCD$ *if* $A = 4A_2$*? Prove your claim.*

Exercise 2. *Suppose that* A *and* B *are distinct* $n \times n$ *matrices such that:*

i) $A^3 = B^3$
ii) $A^2 \cdot B = B^2 \cdot A$.

Prove that the matrix $A^2 + B^2$ *is not invertible.*

Exercise 3. *a) Prove that for all* $n \in \mathbb{N}$ *the following identity holds:*

$$1^3 + 2^3 + 3^3 + \cdots n^3 = \Big(\frac{n(n+1)}{2}\Big)^2.$$

b) Suppose that $n \in \mathbb{N}$ *is a positive integer and suppose that* $a_1, a_2, \ldots, a_n$ *are mutually distinct positive integers. Prove that:*

$$\Big(\sum_{j=1}^{n} a_j^5\Big) + \Big(\sum_{j=1}^{n} a_j^7\Big) \geq 2\Big(\sum_{j=1}^{n} a_j^3\Big)^2$$

c) When does equality hold in part b)?

Exercise 4. *Let the sequence* $(a_n)_{n\geq 0}$ *be defined as follows:*

i) $a_0 := 0, a_1 := 1$.
ii) *Given* $a_0, a_1, a_2, \ldots, a_n$*, the term* a_{n+1} *is defined to be the smallest non-negative integer such that there don't exist* $i, j \in \{0, 1, \ldots, n\}$*, with* $i \leq j$ *such that* a_i, a_j, a_{n+1} *are three consecutive terms of an arithmetic sequence, i.e.* $a_i + a_{n+1} = 2a_j$.

a) Find a_2, a_3 *and* a_4.
b) Prove that a_n *equals the* n*-th non-negative integer whose expansion in base* 3 *doesn't contain the digit* 2.
c) Find a_{100}.