1. Let $u(x, y)$ solve the wave equation

$$u_{tt} = 4u_{xx}, \quad 0 \leq x \leq \pi,\ t \geq 0$$

satisfying the boundary conditions

$$u(0, t) = 0$$
$$u(\pi, t) = 0$$

and initial conditions

$$u(x, 0) = \sin(x) - 2\sin(3x)$$
$$u_t(x, 0) = 0$$

What is $u(\frac{\pi}{2}, \frac{\pi}{2})$?

A: -3	B: -1	C:1	D: 3
E: 0	F: $-\pi$	G: π	H: 3π

Separate variables: let $u(x, t) = \phi(x)g(t)$. Separating gives us

$$\phi'' + \lambda\phi = 0 \quad \phi(0) = \phi(\pi) = 0$$
$$g'' + 4\lambda g = 0 \quad g'(0) = 0.$$

The answer to the first problem is $\lambda = n^2$, $\phi(x) = \sin(nx)$; with this value of λ, we get $g(t) = \cos(2nt)$. Then product solutions are $\phi(x)g(t) = \sin(nx)\cos(2nt)$, so the general solution is

$$u(x, t) = \sum_{n=1}^{\infty} A_n \sin(nx)\cos(2nt).$$

To get the coefficients, we use the initial conditions

$$u(x, 0) = \sum_{n=1}^{\infty} A_n \sin(nx) = \sin(x) - 2\sin(3x),$$

so $A_1 = 1$, $A_3 = -2$, and all other $A_n = 0$. Thus,

$$u(x, t) = \sin(x)\cos(2t) - 2\sin(3x)\cos(6t),$$

and

$$u(\frac{\pi}{2}, \frac{\pi}{2}) = \sin(\frac{\pi}{2})\cos(\pi) - 2\sin(\frac{3\pi}{2})\cos(3\pi) = -3,$$

so the answer is A, -3.

2. The displacement $u(x,y,t)$ of a vibrating rectangular membrane satisfies the PDE

$$u_{tt} = c^2(u_{xx} + u_{yy}) \quad 0 \le x \le \pi,\ 0 \le y \le \frac{\pi}{4},\ t \ge 0$$

with BCs

$$\begin{array}{ll} u(0,y,t) = 0 & u(\pi,y,t) = 0 \\ u(x,0,t) = 0 & \frac{\partial u}{\partial y}(x,\frac{\pi}{4},t) = 0. \end{array}$$

What is the smallest natural frequency of the membrane? (This can be done without fully solving for u.)

A: c	B: $2c$	C: $c\sqrt{2}$	D: $c\sqrt{3}$
E: $c\sqrt{5}$	F: $c\sqrt{8}$	G: $c\sqrt{\pi}$	H: 0

Separate $u(x,y,t) = \phi(x,y)h(t)$; then

$$\begin{aligned} &\Delta\phi + \lambda\phi = 0 \\ &h'' + c^2\lambda h = 0. \end{aligned}$$

Now we separate ϕ: $\phi(x,y) = f(x)g(y)$:

$$\frac{f''}{f} = -\frac{g''}{g} - \lambda = \mu,$$

so

$$\begin{aligned} &f'' + \mu f = 0 \quad f(0) = f(\pi) \\ &g'' + (\lambda - \mu)g = 0 \quad g(0) = g'(\frac{\pi}{4}) = 0. \end{aligned}$$

For f, we have $\mu = m^2$, and $f(x) = \sin(nx)$. For g, we get

$$g(y) = c_1 \sin((\sqrt{\lambda - \mu})y) + c_2 \cos((\sqrt{\lambda - \mu})y);$$

the boundary conditions $g(0) = 0$ gives $c_2 = 0$ and $g'(\frac{\pi}{4}) = 0$ gives $c_1(\sqrt{\lambda - \mu})\cos((\sqrt{\lambda - \mu})\frac{\pi}{4})$; so we need $(\sqrt{\lambda - \mu})\frac{\pi}{4} = \frac{2n-1}{2}$. Thus,

$$\lambda - \mu = (4n - 2)^2.$$

So $\lambda_{mn} = m^2 + (4n-2)^2$. Solving for $h(t)$, we get $h(t) = c_1 \cos(c\sqrt{\lambda_{mn}}t) + c_2 \sin(c\sqrt{\lambda_{mn}}t)$. The natural frequency is $c\sqrt{\lambda_{mn}}$; the smallest is $c\sqrt{\lambda_{11}} = c\sqrt{5}$. So the answer is E.

3. Solve the heat equation

$$u_t = k\Delta u$$

on a semicircle $0 \le r \le a$, $0 \le \theta \le \pi$ with boundary conditions

$$u(r,0,t) = 0,\ u(r,\pi,t) = 0,\ u(a,\theta,t) = 0$$

and initial conditions

$$u(r,\theta,0) = f(r,\theta).$$

Separating $u(r,\theta,t) = p(x)q(\theta)h(t)$ gives (eventually)

$$\begin{aligned}
&r^2p'' + rp' + (\lambda r^2 - \mu)p = 0 \quad p(a) = 0, |p(0)| < \infty \\
&q'' + q = 0 \quad q(0) = q(\pi) = 0 \\
&h' + \lambda k h = 0.
\end{aligned}$$

Then $q(\theta) = \sin(m\theta)$, and $\mu = m^2$. Solving for p, we get $p(r) = J_m(\sqrt{\lambda}r)$ (this was on the formula sheet). We use the boundary condition to get λ: $p(a) = J_m(\sqrt{\lambda}a) = 0$. If z_{mn} are the positive zeroes of J_m, then we want $\sqrt{\lambda}a = z_{mn}$, so our eigenvalues are

$$\lambda_{mn} = \left(\frac{z_{mn}}{a}\right)^2.$$

Finally, solving for h gives $h(t) = e^{-\lambda_{mn}kt}$. Then the solution is given by

$$u(x,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn} J_m(\sqrt{\lambda_{mn}}r)\sin(m\theta)e^{-\lambda_{mn}kt}.$$

Plugging in ICs,

$$u(x,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn} J_m(\sqrt{\lambda_{mn}}r)\sin(m\theta) = f(r,\theta)$$

tells us the coefficients are

$$B_{mn} = \frac{\int_0^{\pi}\int_0^{a} f(r,\theta) J_m(\sqrt{\lambda_{mn}}r)\sin(m\theta) r\,dr\,d\theta}{\int_0^{\pi}\int_0^{a} J_m(\sqrt{\lambda_{mn}}r)^2 \sin(m\theta)^2 r\,dr\,d\theta}.$$

4. Consider the eigenvalue problem

$$\phi'' + 2x\phi' + \frac{\lambda}{x}\phi = 0 \quad \phi(1) = \phi(2) = 0.$$

Express this problem in standard Sturm-Liouville form. If $\phi_1, \phi_2, \ldots$ are the eigenfunctions of this problem, and

$$f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x),$$

write a formula for a_n (in terms of f and ϕ_n).

Multiplying the equation by $H(x)$, we get

$$H(x)\phi'' + 2xH(x)\phi' + \lambda\frac{H(x)}{x}\phi = 0;$$

we want this to look like

$$\frac{d}{dx}(p(x)\phi') + (\lambda\sigma(x) + q(x))\phi = 0,$$

so we need $H(x) = p(x)$ and $2xH(x) = p'(x)$. Thus, we have $\frac{p'}{p} = 2x$. Integrating both sides gives $\ln p(x) = x^2$, so $p(x) = e^{x^2}$. Then our equation is

$$e^{x^2}\phi'' + 2xe^{x^2}\phi' + \lambda\frac{e^{x^2}}{x}\phi = 0;$$

that is,

$$\frac{d}{dx}(e^{x^2}\phi') + \lambda\frac{e^{x^2}}{x}\phi = 0.$$

The coefficient a_n is given by the quotient

$$a_n = \frac{\int_1^2 f(x)\phi_n(x)\frac{e^{x^2}}{x}\,dx}{\int_1^2 \phi_n(x)^2\frac{e^{x^2}}{x}\,dx}.$$

Note that the limits are taken from the boundary conditions $\phi(1) = \phi(2) = 0$.

5. Consider the problem

$$\begin{aligned}
&u_t = ku_{xx} \quad 0 \le x \le 2, \quad t \ge 0 \\
&u(0,t) = \cos t \\
&\frac{\partial u}{\partial x}(2,t) = e^{-t} \\
&u(x,0) = 0.
\end{aligned}$$

Find a reference temperature $r(x,t)$ such that $v = u - r$ satisfies homogeneous boundary conditions. v satisfies a PDE of the form $v_t = v_{xx} + \bar{Q}(x,t)$. What is $\bar{Q}(x,t)$?

There are many potential choices for $r(x,t)$. The simplest is probably

$$r(x,t) = \cos t + xe^{-t}.$$

Note that we indeed have $r(0,t) = \cos t$, and $r_x(x,t) = e^{-t}$, so in particular $r_x(2,t) = e^{-t}$.

Then $u = v + r$. We plug that into the PDE:

$$(v+r)_t = k(v+r)_{xx}.$$

Rearranging, we get

$$\begin{aligned}
v_t &= kv_{xx} + (kr_{xx} - r_t) \\
&= kv_{xx} + k\frac{\partial^2}{\partial x^2}(\cos t + xe^{-t}) - \frac{\partial}{\partial t}(\cos t + xe^{-t}) \\
&= kv_{xx} + (\sin t + xe^{-t}).
\end{aligned}$$

Thus (for this choice of r), we have $\bar{Q} = \sin t + xe^{-t}$.