On the purity conjecture of Nisnevich for torsors under reductive group schemes

Roman Fedorov

University of Pittsburgh

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References and the plan

- ArXiv:2109.10332
- These slides:
  - pitt.edu/~fedorov/NisnevichConjSlides.pdf

Plan:

- Formulation of Nisnevich’s purity Conjecture
- Proof of Nisnevich’s purity Conjecture (under some conditions)
- Abstract formulation of the Grothendieck–Serre and Nisnevich’s purity Conjectures.
- Counterexamples to Nisnevich’s purity Conjecture.
**Conjecture**

Let $R$ be a regular semilocal integral domain. Let $G$ be a reductive group scheme over $R$. Let $E$ be a $G$-torsor. If the restriction of $E$ to the fraction field of $R$ is trivial, then $E$ is trivial.

**Known cases**

- $R$ is a DVR.
- $G$ is a torus.
- $R$ contains a field.
- $R$ is unramified over a DVR, $G$ is quasisplit.
Nisnevich purity conjecture

**Conjecture**

Let $R$ be a regular semilocal integral domain. Let $G$ be a reductive group scheme over $R$. Let $f \in R$ be such that for all maximal ideals $m \subset R$ we have $f \notin m^2$. Let $E$ be a $G$-torsor over $R_f$. If the restriction of $E$ to the fraction field of $R$ is trivial, then $E$ is trivial.

Previously known cases

- $G = \text{GL}_n$, $R$ is local and contains a field. (Bhatwadekar–Rao’83, Popescu’02). And this is non-trivial!
- $\dim R = 1$ (reduces to Grothendieck–Serre).
- $R$ is local of dimension 2, the residue field is infinite, $G$ is quasi-split (Nisnevich’89).

False unless $G$ satisfies an isotropy condition.
Let $G$ be a semisimple group scheme over a connected scheme $U$. There is a sequence $U_1, \ldots, U_r$ of finite étale connected $U$-schemes such that

$$G^{\text{ad}} \simeq \prod_{i=1}^r G^i,$$

where $G^i$ is the Weil restriction of a simple $U_i$-group scheme. Note that the group schemes $G^i$ are uniquely defined by $G$ up to isomorphism. Recall that a semisimple group scheme $G$ over a scheme $S$ is called isotropic, if it contains a non-trivial split torus. If $S$ is connected and semilocal, $G$ is isotropic if and only if it contains a proper parabolic subgroup scheme.

**Definition**

We say that a semisimple $U$-group scheme $G$ is strongly locally isotropic if each factor $G^i$ of $G^{\text{ad}}$ is isotropic Zariski locally over $U$. 

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Nisnevich Conjecture
Grothendieck–Serre conjecture for families

**Theorem (Panin–Stavrova–Vavilov’15, Fedorov’22)**

Let $R$ be a semilocal integral domain geometrically regular over a field $k$; denote by $K$ the fraction field of $R$. Let $G$ be a reductive group scheme over $R$ such that its adjoint group scheme $G^{\text{ad}}$ is strongly locally isotropic. Let $A$ be a $k$-algebra. Let $E$ be a $G$-torsor over $R \otimes_k A$. If the restriction of $E$ to $K \times_k A$ is trivial, then $E$ is trivial.
The main theorem

Theorem

Let $R$ be a regular semilocal integral domain containing a field $k$. Assume that $k$ is infinite or that the inclusion homomorphism $k \rightarrow R$ admits a left inverse. Let $G$ be a reductive group scheme over $R$ such that $G^{\text{ad}}$ is strongly locally isotropic. Let $f \in R$ be such that for all maximal ideals $m \subset R$ we have $f \notin m^2$. Let $E$ be a $G$-torsor over $R_f$. If the restriction of $E$ to the fraction field of $R$ is trivial, then $E$ is trivial.

By Popescu’s Theorem, we may assume that $R = \mathcal{O}_{X,x}$, where $X$ is an integral smooth affine scheme over a field $k$, $x$ is a finite set of closed points of $X$. We may also assume that the group scheme $G$ is defined over $X$. Finally, we may assume that $f$ can be extended to a function on $X$. 
Main reductions

**Proposition**

Let $X$ be an integral affine scheme smooth over a field $k$, $x$ be a finite set of closed points $X$, and set $R := \mathcal{O}_{X,x}$. Let $G$ be a reductive group scheme over $X$. Let $f \in k[X]$ be a non-zero function such that the hypersurface $\{f = 0\}$ is smooth over $k$. Let $\mathcal{E}$ be a generically trivial $G$-torsor over $R_f$. Then there are

- A section $s \in \mathbb{A}^1_R(R)$;
- $R$-finite closed subschemes $Y \subset Z \subset \mathbb{A}^1_R$ such that $Y$ is $R$-étale and $f|_{s^{-1}(Y)} = 0$;
- an element

$$\mathcal{E}' \in \text{Ker}\left(H^1(\mathbb{A}^1_R - Y, G) \to H^1(\mathbb{A}^1_R - Z, G)\right),$$

such that $(s|_{R_f})^* \mathcal{E}' = \mathcal{E}$. 
Proposition

Let $R$ be a regular semilocal integral domain containing a field $k$. Assume that $k$ is infinite or that the inclusion homomorphism $k \to R$ admits a left inverse. Let $G$ be a reductive group scheme over $R$ such that its adjoint group scheme $G^{ad}$ is strongly locally isotropic. Let $Y \subset Z \subset \mathbb{A}^1_R$ be closed subschemes finite over $R$ such that $Y$ is étale over $R$. Let $E$ be a $G$-torsor over $\mathbb{A}^1_R - Y$ whose restriction to $\mathbb{A}^1_R - Z$ is trivial. Then $E$ is trivial.
Case 1: \( Y \) is “constant”, that is, of the form \( Y_0 \times_k \text{Spec} \ R \). We apply Grothendieck–Serre for families: it is enough to check that \( \mathcal{E} \) is trivial over \( \mathbb{A}^1_K - Y_K \). However, a generically trivial torsor over a Zariski open subset of \( \mathbb{A}^1_K \) is trivial.
Proof of the proposition

Case 2: $Y$ decomposes as $\bigsqcup_{i=1}^{d} Y_i$, where for each $i$ the restriction of the projection $\mathbb{A}_R^1 \to \text{Spec } R$ to $Y_i$ is an isomorphism. Because of our assumption on $R$ we have $d \leq |k|$. Let $t_1, \ldots, t_d$ be distinct elements of $k$. We construct a closed $R$-embedding $\iota: Z_{(2)} \hookrightarrow \mathbb{A}_R^1$ such that $\iota(Y_i) = t_i \times_k \text{Spec } R$. We get an elementary distinguished square

$$
\begin{array}{ccc}
W - Z & \longrightarrow & W - Y \\
\downarrow & & \downarrow \\
A_R^1 - Z' & \longrightarrow & A_R^1 - Y_0 \times_k \text{Spec } R,
\end{array}
$$

where $W$ is a Zariski neighborhood of $Z$. We use this square to descend $E$ onto $A_R^1 - Y_0 \times_k \text{Spec } R$. By case 1 the descended torsor is trivial. Thus $E$ is trivial on $W - Y$.
Proof of the proposition

General case: define $d(Y/\text{Spec } R)$ as the difference between $\deg(Y/\text{Spec } R)$ and the number of connected components of $Y$. We induct on $d(Y/\text{Spec } R)$. The case $d(Y/\text{Spec } R) = 0$ is exactly Case 2. Now we prove the induction step.
Proof of the proposition

Let $V$ be a connected component of $Y$ of degree at least two, we pullback the whole picture along $V \to \text{Spec } R$: $Y' := Y \times_R V$, $Z' := Z \times_R V$, $\mathcal{E}' := \mathcal{E}|_{\mathbb{A}^1_Y - Y'}$. Let $\Delta \subset V \times_R V \subset Y'$ be the "diagonal component"; we have $d(Y'/V) < d(Y/\text{Spec } R)$. By the induction hypothesis $\mathcal{E}'$ is trivial.

The étale morphism $\phi: \mathbb{A}^1_V \to \mathbb{A}^1_R$ maps $\Delta$ isomorphically onto $V$. Let $W$ be a Zariski neighborhood of $\Delta$ in $\mathbb{A}^1_V$ such that $\phi^{-1}(V) \cap W = \Delta$. Then the square

$$
\begin{array}{ccc}
W - Y' & \longrightarrow & W - (Y' - \Delta) \\
downarrow & & \downarrow \\
\mathbb{A}^1_R - Y & \longrightarrow & \mathbb{A}^1_R - (Y - V)
\end{array}
$$

is an elementary distinguished square. We use it to extend $\mathcal{E}$ to $\mathbb{A}^1_R - (Y - V)$. Now apply the induction hypothesis.
Definition

Let $X$ be a Noetherian scheme. We say that a Noetherian $X$-scheme $Y$ is *essentially smooth* over $X$ if it is a filtered inverse limit of smooth $X$-schemes with transition morphisms being open affine embeddings.

We denote by $\text{Sm}_X'$ the full subcategory of the category of $X$-schemes whose objects are Noetherian schemes essentially smooth over $X$.

An important example of an essentially smooth scheme is the following: let $X$ be an affine scheme, let $x$ be a finite set of schematic points of $X$, let $\mathcal{O}_{X,x}$ be the semilocal ring of $x$, then $\text{Spec} \mathcal{O}_{X,x}$ is an object of $\text{Sm}_X'$. 

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Nisnevich Conjecture
Lemma

(i) Let $X$, $Y$, and $Z$ be Noetherian schemes such that $Y$ be essentially smooth over $X$ and $Z$ be essentially smooth over $Y$. Then $Z$ is essentially smooth over $X$.

(ii) A base change of an essentially smooth morphism is essentially smooth.
Definition of Nisnevich semisheaf

An elementary distinguished square is a Cartesian diagram of schemes:

\[
\begin{array}{ccc}
U \times_X V & \longrightarrow & V \\
\downarrow & & \downarrow^p \\
U & \longrightarrow & X,
\end{array}
\]

where \( p \) is étale, \( j \) is an open embedding, and \( p^{-1}(X - U) \rightarrow X - U \) is an isomorphism.

Let \( X \) be a Noetherian scheme and \( F \) be a presheaf of pointed sets on \( Sm'/X \). We say that \( F \) is a Nisnevich semisheaf of pointed sets if for any elementary distinguished square in \( Sm'/X \) the corresponding map

\[
F(X) \rightarrow F(U) \times_{F(U \times_X V)} F(V)
\]

is surjective.
We will formulate the properties under which $F$ satisfies a Grothendieck–Serre type statement.

**(Lim)** $F$ commutes with filtered inverse limits, provided that the transition morphisms are open embeddings.

**(Sec)** Let $U \in \text{Ob}(Sm'/X)$ be an affine integral semilocal scheme such that all its closed points are finite over $k$. Assume that $Z$ is a closed subscheme of $\mathbb{A}^1_U$ finite over $U$. Let $E \in \ker(F(\mathbb{A}^1_U) \to F(\mathbb{A}^1_U - Z))$,

then for every section $\Delta : U \to \mathbb{A}^1_U$ of the projection $\mathbb{A}^1_U \to U$ we have $\Delta^*E = *$. 
(LT) Assume that $W$ is an affine integral semilocal $k$-scheme such that all its closed points are finite over $k$, and let $U \subset W$ be a closed subscheme. Assume that we are given two essentially smooth $k$-morphisms $p_1, p_2 : W \to X$ such that $p_1|_U = p_2|_U$, and this morphism is essentially smooth. Then there is a finite étale $k$-morphism $\pi : W' \to W$ with a section $\Delta : U \to W'$ and an isomorphism of presheaves $(p_1 \circ \pi)^*F \cong (p_2 \circ \pi)^*F$ restricting to the identity isomorphism on $U$.

This property requires some explanation. First of all, $p_i \circ \pi$ are essentially smooth by the lemma, so the pullbacks of presheaves make sense (again, by the lemma). The isomorphism amounts to compatible bijections for any essentially smooth $\psi : T \to W'$:

$$F(T, p_1 \circ \pi \circ \psi) \cong F(T, p_2 \circ \pi \circ \psi).$$
Finally, the condition that this isomorphism restricts to the identity on $U$ amounts to having for every $\psi$ as above a commutative diagram

$$
\begin{array}{ccc}
F(T, p_1 \circ \pi \circ \psi) & \xrightarrow{\sim} & F(T, p_2 \circ \pi \circ \psi) \\
\downarrow & & \downarrow \\
F(T \times_{W} U, p_1 \circ \pi \circ \psi \circ pr_1) & \cong & F(T \times_{W} U, p_2 \circ \pi \circ \psi \circ pr_1),
\end{array}
$$

where $pr_1$ denotes the projection $T \times_{W} U \rightarrow T$. Here the bottom identification comes from the fact that both morphisms are equal:

$$p_i \circ \pi \circ \psi \circ pr_1 = p_i \circ \pi \circ \Delta \circ pr_2 = (p_i|_U) \circ pr_2.$$

(In particular, this morphism is essentially smooth.)
**Theorem**

Let $X$ be a smooth integral affine $k$-scheme, where $k$ is a field. Let $F$ be a Nisnevich semisheaf on $Sm'/X$ satisfying properties (Lim), (LT), and (Sec) above. Then for any finite set of schematic points $y \subset X$ the map $F(\mathcal{O}_{X,y}) \to F(K(X))$, where $K(X)$ is the fraction field of $X$, has a trivial kernel.

Goes back to Colliot–Thélène and Ojanguren’92, [“Espaces principaux homogènes localement triviaux”, Thm 1.1] if $G$ comes from the field.
In the notation of the theorem assume that the Nisnevich semisheaf $F$ satisfies properties (Lim), (LT), and a property stronger than (Sec):

**($\text{SecF}$)** Let $U \in \text{Ob}(Sm'/X)$ be an integral affine semilocal scheme such that all its closed points are finite over $k$. Assume that $Z$ is a closed subscheme of $\mathbb{A}^1_U$ finite over $U$. Let $Y \in \text{Ob}(Sm'/k)$ be an affine scheme, and

$$\mathcal{E} \in \text{Ker} \left( F(\mathbb{A}^1_U \times_k Y) \to F((\mathbb{A}^1_U - Z) \times_k Y) \right),$$

then for every section $\Delta : U \times_k Y \to \mathbb{A}^1_U \times_k Y$ of the projection $\mathbb{A}^1_U \times_k Y \to U \times_k Y$ we have $\Delta^* \mathcal{E} = *$.

Then for any finite set of schematic points $y \subset X$ and any affine scheme $Y \in \text{Ob}(Sm'/k)$ the map

$$F(O_{X,y} \otimes_k k[Y]) \to F(K(X) \otimes_k k[Y]),$$

where $K(X)$ is the fraction field of $X$, has a trivial kernel.
Let \( X \) be a smooth integral affine \( k \)-scheme, where \( k \) is a field. Let \( F \) be a Nisnevich semisheaf on \( \text{Sm}'/X \) satisfying properties (Lim), (LT), and (SecF) above as well as:

\( \text{(A1F)} \) Let \( K \) be the function field of \( X \). Then the map \( F(K[t]) \rightarrow F(K(t)) \) has a trivial kernel.

Let \( y \subset X \) be a finite set of schematic points, let \( f \in k[X] \) be such that for any \( y \in y \) we have \( f \notin m_y^2 \). Assume additionally that \( k \) is infinite or one of the points \( y \in y \) is \( k \)-rational. Then the map

\[
F((\mathcal{O}_{X,y})_f) \rightarrow F(K)
\]

has a trivial kernel.
Counterexamples

Theorem

There is a regular local ring $R$ containing an infinite field, a simple simply-connected group scheme $G$ over $R$, and a generically trivial $G$-torsor $E$ over $\mathbb{A}^1_R$ that cannot be extended to $\mathbb{P}^1_R$.

This contradicts Nisnevich’s Conjecture for $R[t]_0$. The theorem is derived from the proposition:

Proposition

Let $B$ be an integral Noetherian normal local ring that is not a field. Assume that $B$ contains an infinite field. Let $G$ be a semisimple $B$-group scheme anisotropic over the fraction field of $B$. Then there is a maximal ideal $\mathfrak{m} \subset B[t]$ and a generically trivial $G$-torsor $E$ over $\mathbb{A}^1_{B[t]}$ that cannot be extended to $\mathbb{P}^1_{B[t]}$.  

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Nisnevich Conjecture
Preparation for the proof

- There is a finite étale cover $\text{Spec } B' \to \text{Spec } B$ and a $k$-group scheme $G$ such that $G_{B'} \cong G \times_k \text{Spec } B'$.
- We construct a closed $B$-embedding $\text{Spec } B' \to \mathbb{A}^1_B - (0 \times \text{Spec } B)$.

\[
\text{Mor}_k(\mathbb{A}^1_{B'}, \text{Gr}_G) = \left\{ (\mathcal{E}, \tau), \mathcal{E} \text{ is a } G \text{-torsor over } \mathbb{A}^1_k \times \mathbb{P}^1_B, \tau \text{ is a trivialization on } \mathbb{A}^1_k \times (\mathbb{P}^1_B - \text{Spec } B') \right\}.
\]

And a similar statement is true with $\mathbb{A}^1$ replaced with $\mathbb{P}^1$.
(Here $\text{Gr}_G$ stands for the affine Grassmannian, see [Fedorov, “Affine Grassmannians of group schemes and exotic principal bundles over $\mathbb{A}^1$”].)
The construction

- Construct a morphism $\mathbb{A}_B^1 \to Gr_G$ that cannot be extended to $\mathbb{P}_B^1$.
- Use the previous bijection to get a $G$-torsor over $\mathbb{A}_k^1 \times \mathbb{P}_B^1$, with a trivialization $\tau$ on $\mathbb{A}_k^1 \times (\mathbb{P}_B^1 - \text{Spec } B')$.
- It cannot be extended to a torsor over $\mathbb{P}_k^1 \times \mathbb{P}_B^1$, with a trivialization on $\mathbb{P}_k^1 \times (\mathbb{P}_B^1 - \text{Spec } B')$.
- It remains to show that there is a maximal ideal $m \subset B[t]$ such that the restriction of $\mathcal{E}$ to $\mathbb{A}_k^1 \times \text{Spec } B[t]_m$ cannot be extended to $\mathbb{P}_k^1 \times \text{Spec } B[t]_m$. 
The last item is accomplished via the following proposition, which follows easily from results of [Fedorov, “Affine Grassmannians of group schemes and exotic principal bundles over $\mathbb{A}^1$”].

**Proposition**

(i) Let $X$ be an integral Noetherian normal $k$-scheme. Assume that $H \to X$ is a semisimple group scheme such that $H_k(X)$ is anisotropic. Let $E$ be a generically trivial $H$-torsor over $\mathbb{A}^1_k \times X$. Then $E$ can be extended to $\mathbb{P}^1_k \times X$ if and only if for all closed points $x \in X$ the restriction of $E$ to $\mathbb{A}^1_k \times \text{Spec} \mathcal{O}_{X,x}$ can be extended to $\mathbb{P}^1_k \times \text{Spec} \mathcal{O}_{X,x}$.

(ii) The extension from part (i) is unique (if it exists) in the following sense. Let $E_1$ and $E_2$ be $H$-torsors over $\mathbb{P}^1_k \times X$ and let $\tau_i : E \to E_i|_{\mathbb{A}^1_k \times X}$ be isomorphisms. Then there is a unique isomorphism of $H$-torsors $\phi : E_1 \to E_2$ such that $\phi|_{\mathbb{A}^1_k \times X} = \tau_2 \circ \tau_1^{-1}$. 

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Nisnevich Conjecture