1. Let $X$ be a complex manifold of dimension $n$ and let $D \subset X$ be a smooth hypersurface. Let $\Omega^p_X(*D)$ denote the sheaf of meromorphic $k$-forms on $X$ that are holomorphic on $X - D$. Consider the subsheaf

$$\Omega^p_X(\log D) = \left\{ \alpha \in \Omega^p_X(*D) \mid \text{both } \alpha \text{ and } d\alpha \text{ have simple poles along } D \right\}.$$

(i) Show that $\Omega^p_X(\log D)$ is a locally free sheaf of rank $\binom{n}{p}$. Concretely show the following:

(a) If $U \subset X - D$ is open, then $\Omega^p_X(\log D)|_U \subset \Omega^p_U$ as sheaves of $\mathcal{O}_U$ modules.

(b) Near $D$, for any small enough open $U$ choose local holomorphic coordinates $z_1, z_2, \ldots, z_n$ so that $D$ is given by the equation $z_1 = 0$. Show that the forms

$$\left\{ dz_{i_1} \wedge dz_{i_2} \cdots \wedge dz_{i_p}, \frac{1}{z_1} dz_{i_1} \wedge dz_{i_2} \cdots \wedge dz_{i_p} \right\}_{1 < i_1 < i_2 < \ldots < i_p}$$

give a basis for $\Omega^p_X(\log D)|_U$ as an $\mathcal{O}_U$ module.

(ii) Let $i : D \hookrightarrow X$ be the inclusion map. Consider the Poincare residue map

$$\text{Res} : \Omega^p_X(\log D) \xrightarrow{i^*} i_\ast \Omega^{k-1}_D$$

$$\gamma \wedge \frac{dz_1}{z_1} \xrightarrow{\text{Res}} \gamma|_D.$$

Show that the Poincare residue map fits in a short exact sequence of sheaves

$$0 \rightarrow \Omega^p_X(\log D) \rightarrow \Omega^p_X \rightarrow \Omega^p_X(\log D) \rightarrow 0.$$ (1)

(iii) Let $\mathcal{O}_X(-D)$ be the ideal sheaf of $D$. Use (i) to show that

$$\Omega^p_X(\log D) \otimes \mathcal{O}_X(-D) = \ker [\Omega^p_X \rightarrow i_\ast \Omega^p_D].$$

This gives a short exact sequence of sheaves

$$0 \rightarrow \Omega^p_X(\log D) \otimes \mathcal{O}_X(-D) \rightarrow \Omega^p_X \rightarrow i_\ast \Omega^p_D \rightarrow 0,$$
or equivalently a short exact sequence

\[0 \longrightarrow \Omega^p_X (\log D) \longrightarrow \Omega^p_X (D) \longrightarrow i_* \Omega^p_D (D) \longrightarrow 0.\]

Show that the sequences (1) and (2) fit in a commutative diagram:

\[
\begin{array}{ccccccc}
0 & 0 & i_* \Omega^p_D (D) & \longrightarrow & i_* \Omega^p_D (D) & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \\
0 & \Omega^p_X & \Omega^p_X (D) & \longrightarrow & i_* \Omega^p_X (D) & \longrightarrow & 0 \\
\| & & \| & & \| & & \\
0 & \Omega^p_X & \Omega^p_X (\log D) & \longrightarrow & i_* \Omega^p_D (\log D) & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \\
0 & 0 & 0 & & 0 & & \\
\end{array}
\]

2. Prove Bott’s vanishing theorem on \(\mathbb{P}^n\):

\[H^q (\mathbb{P}^n, \Omega^p \otimes \mathcal{O}(k)) = 0\]

unless

\[
\begin{cases}
q = 0 \text{ or } n \\
p = q, \text{ and } k = 0.
\end{cases}
\]

Argue by induction on \(n\).

(i) Prove the statement for \(n = 1\) using Serre duality and the Kodaira vanishing theorem.

(ii) Prove that Bott’s vanishing for \(\mathbb{P}^{n-1}\) implies Bott’s vanishing for \(\mathbb{P}^n\):

(a) Show that the statement of Bott’s vanishing for \(k \geq 0\) implies the statement for all \(k\).

(b) Use Hodge theory to show that Bott’s vanishing holds for \(k = 0\).

(c) Set \(X = \mathbb{P}^n\) and let \(\mathbb{P}^{n-1} \cong D \subset X\) be a hyperplane. Show that in the long exact sequence in cohomology associated to the residue sequence (1), the edge map

\[\delta : H^{p-1} (D, \Omega^p_{D}) \rightarrow H^{p} (X, \Omega^p_X)\]

is an isomorphism as follows. Let \(\omega \in A^{1,1}(X)\) be the Fubini-Studi form and let \(\omega_D \in A^{1,1}(D)\) be its restriction to the hyperplane \(D\). Then \(\omega^p\) is a Dolbeault representative of a non-zero vector in \(H^p(X, \Omega^p_X) \cong \mathbb{C}\) and \(\omega^p_D\) is a Dolbeault representative of a non-zero vector in \(H^p(D, \Omega^p_D) \cong \mathbb{C}\). Let \(h\) be the Hermitian metric on \(\mathcal{O}_X(1)\) for which the Chern connection has curvature \(\omega\). Let \(s \in H^0(X, \mathcal{O}_X(1))\) be a holomorphic section that vanishes on \(D\).
* Show that the \((p, p-1)\)-form \(\alpha := (\partial \log ||s||_h) \wedge \omega^{p-1}\) is a section in \(\mathcal{A}_{X}^{p,0}(\Omega_X^p(\log D))\) and that \(\text{Res}(\alpha) = \omega_D^{p-1}\).

* Show that \(\delta([\omega_D^{p-1}]) = [\omega^p]\).

(d) Show that (b), (c) and Bott’s vanishing for \(\mathbb{P}^{n-1}\) imply Bott’s vanishing for \(\mathbb{P}^n\) and \(k = 1\).

(e) Tensor the short exact sequences (1) and (2) with \(\mathcal{O}(k)\) and use the associated long exact sequences in cohomology to argue that Bott’s vanishing for \(\mathbb{P}^{n-1}\) together with Bott’s vanishing for \(\mathbb{P}^n\) and some \(k > 0\) imply Bott’s vanishing for \(\mathbb{P}^n\) and \(k + 1\).