# Math 6220/6230, Homework 9

**1.** Let $X$ be a complex manifold of dimension $n$ and let $D \subset X$ be a smooth hypersurface. Let $\Omega^p_X(*D)$ denote the sheaf of meromorphic $k$-forms on $X$ that are holomorphic on $X - D$. Consider the subsheaf

$$\Omega^p_X(\log D) = \left\{ \alpha \in \Omega^p_X(*D) \;\middle|\; \begin{array}{l} \text{both } \alpha \text{ and } d\alpha \text{ have} \\ \text{simple poles along } D. \end{array} \right\}$$

(i) Show that $\Omega^p_X(\log D)$ is a locally free sheaf of rank $\binom{n}{p}$. Concretely show the following:

(a) If $U \subset X - D$ is open, then $\Omega^p_X(\log D)_{|U} \subset \Omega^p_U$ as sheaves of $\mathcal{O}_U$ modules.

(b) Near $D$, for any small enough open $U$ choose local holomorphic coordinates $z_1, z_2, \ldots, z_n$ so that $D$ is given by the equation $z_1 = 0$. Show that the forms

$$\left\{ dz_{i_1} \wedge dz_{i_2} \ldots \wedge dz_{i_p}, \frac{1}{z_1} dz_1 \wedge dz_{i_2} \ldots \wedge dz_{i_p} \right\}_{1 < i_1 < i_2 < \ldots i_p}$$

give a basis for $\Omega^p_X(\log D)_{|U}$ as an $\mathcal{O}_U$ module.

(ii) Let $i : D \hookrightarrow X$ be the inclusion map. Consider the Poincare residue map

$$\begin{aligned} \mathrm{Res} : \ \Omega^p_X(\log D) &\longrightarrow i_* \Omega^{k-1}_D \\ \gamma \wedge \tfrac{dz_1}{z_1} &\longrightarrow \gamma_{|D}. \end{aligned}$$

Show that the Poincare residue map fits in a short exact sequence of sheaves

$$0 \longrightarrow \Omega^p_X \longrightarrow \Omega^p_X(\log D) \xrightarrow{\mathrm{Res}} i_* \Omega^{k-1}_D \longrightarrow 0. \tag{1}$$

(iii) Let $\mathcal{O}_X(-D)$ be the ideal sheaf of $D$. Use (i) to show that

$$\Omega^p_X(\log D) \otimes \mathcal{O}_X(-D) = \ker \left[ \Omega^p_X \to i_* \Omega^p_D \right].$$

This gives a short exact sequence of sheaves

$$0 \longrightarrow \Omega^p_X(\log D) \otimes \mathcal{O}_X(-D) \longrightarrow \Omega^p_X \longrightarrow i_* \Omega^p_D \longrightarrow 0,$$

or equivalently a short exact sequence

$$0 \longrightarrow \Omega^p_X(\log D) \longrightarrow \Omega^p_X(D) \longrightarrow i_*\Omega^p_D(D) \longrightarrow 0. \tag{2}$$

Show that the sequences (1) and (2) fit in a commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & i_*\Omega^p_D(D) & = & i_*\Omega^p_D(D) & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \to & \Omega^p_X & \longrightarrow & \Omega^p_X(D) & \longrightarrow & i_*\Omega^p_{X|D}(D) & \to & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \to & \Omega^p_X & \to & \Omega^p_X(\log D) & \longrightarrow & i_*\Omega^{p-1}_D & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{3}$$

**2.** Prove *Bott's vanishing theorem* on $\mathbb{P}^n$:

$$H^q(\mathbb{P}^n, \Omega^p \otimes \mathcal{O}(k)) = 0$$

unless

$$\begin{cases} q & = 0 \text{ or } n \\ p & = q, \text{ and } k = 0. \end{cases}$$

Argue by induction on $n$.

(i) Prove the statement for $n = 1$ using Serre duality and the Kodaira vanishing theorem.

(ii) Prove that Bott's vanishing for $\mathbb{P}^{n-1}$ implies Bott's vanishing for $\mathbb{P}^n$:

(a) Show that the statement of Bott's vanishing for $k \geq 0$ implies the statement for all $k$.

(b) Use Hodge theory to show that Bott's vanishing holds for $k = 0$.

(c) Set $X = \mathbb{P}^n$ and let $\mathbb{P}^{n-1} \cong D \subset X$ be a hyperplane. Show that in the long exact sequence in cohomology associated to the residue sequence (1), the edge map

$$\delta : H^{p-1}(D, \Omega^{p-1}_D) \to H^p(X, \Omega^p_X)$$

is an isomorphism as follows. Let $\omega \in A^{1,1}(X)$ be the Fubini-Studi form and let $\omega_D \in A^{1,1}(D)$ be its restriction to the hyperplane $D$. Then $\omega^p$ is a Dolbeault representative of a non-zero vector in $H^p(X, \Omega^p_X) \cong \mathbb{C}$ and $\omega_D^{p-1}$ is a Dolbeault representative of a non-zero vector in $H^{p-1}(X, \Omega^{p-1}_D) \cong \mathbb{C}$. Let $h$ be the Hermitian metric on $\mathcal{O}_X(1)$ for which the Chern connection has curvature $\omega$. Let $s \in H^0(X, \mathcal{O}_X(1))$ be a holomorphic section that vanishes on $D$.

* Show that the $(p, p-1)$-form $\alpha := (\partial \log ||s||_h) \wedge \omega^{p-1}$ is a section in $\mathcal{A}_X^{p,0}(\Omega_X^p(\log D))$ and that $\mathrm{Res}(\alpha) = \omega_D^{p-1}$.
* Show that $\delta([\omega_D^{p-1}]) = [\omega^p]$.

(d) Show that (b), (c) and Bott's vanishing for $\mathbb{P}^{n-1}$ imply Bott's vanishing for $\mathbb{P}^n$ and $k = 1$.

(e) Tensor the short exact sequences (1) and (2) with $\mathcal{O}(k)$ and use the associated long exact sequences in cohomology to argue that Bott's vanishing for $\mathbb{P}^{n-1}$ together with Bott's vanishing for $\mathbb{P}^n$ and some $k > 0$ imply Bott's vanishing for $\mathbb{P}^n$ and $k + 1$.