Math 6220/6230, Homework 8

1. Let X be a complex manifold and let E, F be holomorphic vector bundles on X. An *extension* of E by F is a short exact sequence of holomorphic vector bundles

$$\xi: \ 0 \longrightarrow F \longrightarrow V \longrightarrow E \longrightarrow 0.$$

A *morphism* $\xi \to \eta$ of extensions is a commutative diagram

$$\begin{array}{cccccccccc} \xi: & 0 & \longrightarrow & F & \longrightarrow & V & \longrightarrow & E & \longrightarrow & 0 \\ & & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle b} & & \\ \eta: & 0 & \longrightarrow & F & \longrightarrow & W & \longrightarrow & E & \longrightarrow & 0 \end{array}$$

where a, f and b are morphisms of vector bundles. A morphism of extensions is called a *congruence* if $a = \mathrm{id}_F$ and $b = \mathrm{id}_E$. Let $Ext^1(E, F)$ be the set of congruence classes of extensions of E by F. Given any morphism of vector bundles $\varphi : A \to E$ and an extension ξ of E by F we can construct a new extension

$$\xi \circ \varphi: \ 0 \longrightarrow F \longrightarrow V \times_E A \longrightarrow A \longrightarrow 0.$$

Similarly, given a morphism of vector bundles $\psi : F \to B$ and an extension ξ of E by F we can construct a new extension

$$\psi \circ \xi: \ 0 \longrightarrow B \longrightarrow (V \oplus B)/F \longrightarrow E \longrightarrow 0.$$

(i) Let ξ and η be two extensions of E by F and let $c \in \mathbb{C}$ be a complex number. Define new extensions of E by F as follows:

(Baer sum) $\xi + \eta := \mathbf{sum} \circ (\xi \oplus \eta) \circ \Delta$, where $\xi \oplus \eta$ is the direct sum extension of $E \oplus E$ by $F \oplus F$, $\mathbf{sum} : F \oplus F \to F$ is the sum map $\mathbf{sum}(a \oplus b) := a + b$, and $\Delta : E \to E \oplus E$ is the diagonal map $\Delta(x) := x \oplus x$;

(Rescalling) $c \cdot \xi := \xi \circ (c \cdot \mathrm{id}_E) = (c \cdot \mathrm{id}_F) \circ \xi$.

Show that the Baer sum and the rescalling respect congruences and induce a vector space structure on $Ext^1(E, F)$.

(ii) Show that there is an isomorphism between the vector space $Ext^1(E,F)$ and the cohomology space $R^1\operatorname{Hom}(E,F) = \operatorname{Hom}_{D^b(X)}(E, F[1])$.

(iii) Let $\mathfrak{U}$ be an open cover of X which is acyclic for both E and F. For every $U \in \mathfrak{U}$ consider the split extension

$$\xi_U : 0 \longrightarrow F_{|U} \longrightarrow F_{|U} \oplus E_{|U} \longrightarrow E_{|U} \longrightarrow 0.$$

Let $\sigma = \{\sigma_{UV}\} \in Z^1(\mathfrak{U}, \underline{\operatorname{Hom}}(E,F))$ and let

$$g_{UV}: \ (F \oplus E)_{|U\cup V} \xrightarrow{\cong} (F \oplus E)_{|U \cup V}$$
$$\begin{pmatrix} a \\ b \end{pmatrix} \longrightarrow \begin{pmatrix} \mathrm{id} & \sigma_{UV} \\ 0 & \mathrm{id} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}.$$

Show that $g_{UV} \circ g_{VW} = g_{UW}$ and that $\{g_{UV}\}$ can be used to glue the extensions ξ_U into an extension

$$\xi : \ 0 \longrightarrow F \longrightarrow V \longrightarrow E \longrightarrow 0.$$

Show that the assignment $\sigma \mapsto \xi$ gives an isomorphism of vector spaces $H^1(X, \underline{\operatorname{Hom}}(E,F)) \cong \operatorname{Ext}^1(E,F)$.

2. Use the previous problem and Serre's vanishing theorem to show that a holomorphic vector bundle on $\mathbb{P}^1$ of rank r must be isomorphic to $\mathcal{O}(k_1) \oplus \ldots \oplus \mathcal{O}(k_r)$, for some uniquely determined integers $k_1 \geq k_2 \geq \ldots \geq k_r$.

3. Let n be a positive integer, and let a be an integer with $0 \leq a \leq n$. Let Z be the normalization of the cyclic cover $\mathbb{P}^1\left[\sqrt[n]{x_0^a x_1^{n-a}}\right]$. Describe explicitly all components of Z and the action of the Galois group on each component.