Math 6220/6230, Homework 7

1. Let H be a free abelian group of rank $2g$. Let $(H, F^\bullet H_\mathbb{C})$ be a pure Hodge structure of weight one and let

$$A := \frac{H_\mathbb{C}}{H + F^1 H_\mathbb{C}}$$

(a) Show that A is a g dimensional complex torus.

(b) Show that $(H, F^\bullet H_\mathbb{C})$ and $(H^1(A,\mathbb{Z}), F^\bullet H^1_{dR}(A,\mathbb{C}))$ are dual Hodge structures.

2. Let $H_\mathbb{R}$ be a real vector space and let $H_\mathbb{C} = H_\mathbb{R} \otimes \mathbb{C}$ be its complexification. Let $H_\mathbb{C} = \oplus_{p+q=k} H^{p,q}$ be a direct sum decomposition for which $\overline{H^{p,q}} = H^{q,p}$. For every $z \in \mathbb{C}^\times$ consider the $\mathbb{C}$-linear Weil operator $\rho(z) : H_\mathbb{C} \to H_\mathbb{C}$ which acts on $H^{p,q}$ as multiplication by $z^p \bar{z}^q$.

(a) Show that $\rho : \mathbb{C}^\times \to GL(H_\mathbb{C})$ is a differentiable homomorphism of Lie groups.

(b) Consider the anti-holomorphic involution $\tau : GL(H_\mathbb{C}) \to GL(H_\mathbb{C})$ defined by the rule $\tau(g)(u) = \overline{g(\bar{u})}$. Show that the action ρ satisfies $\rho(\bar{z}) = \tau(\rho(z))$.

(c) Show that $\rho(a) = a^k \cdot \mathrm{id}$ for all $a \in \mathbb{R}^\times$.

3. Let $H_\mathbb{R}$ be a real vector space and let $H_\mathbb{C} = H_\mathbb{R} \otimes \mathbb{C}$ be its complexification. Suppose $\rho : \mathbb{C}^\times \to GL(H_\mathbb{C})$ is a differentiable homomorphism of Lie groups satisfying $\rho(\bar{z}) = \tau(\rho(z))$ for all $z \in \mathbb{C}^\times$ and $\rho(a) = a^k \cdot \mathrm{id}$ for all $a \in \mathbb{R}^\times$.

(a) Show that there exists a character decomposition $H_\mathbb{C} = \oplus_\chi H_\chi$ labeled by differentiable homomorphisms $\chi : \mathbb{C}^\times \to \mathbb{C}^\times$ so that $\rho(z)$ acts as $\chi(z) \cdot \mathrm{id}$ on H_χ.

(b) Show that only the characters $\chi_{p,q}(z) := z^p \bar{z}^q$ with $p + q = k$ appear in this decomposition.

(c) Set $H^{p,q} := H_{\chi_{p,q}}$. Show that $\overline{H^{p,q}} = H^{q,p}$.

4. Let X be a compact Kähler manifold. Consider the complex torus A corresponding (as in problem 1) to the weight one Hodge structure on $H^1(X, \mathbb{Z})$. Show that A is isomorphic to $\ker(\mathrm{Pic}(X) \to H^2(X, \mathbb{Z}))$, where $\mathrm{Pic}(X) \to H^2(X, \mathbb{Z}))$ is the map coming from the exponential sequence. The torus A is called the Picard torus of X and is denoted by $\mathrm{Pic}^0(X)$.