Math 6220, Homework 6

1. Let X be a complex manifold. Define the category $(\mathsf{Vect/X})$ whose objects are holomorphic vector bundles on X and whose morphisms[1] are the holomorphic maps of total spaces which are linear on the fibers. Given a vector bundle $\pi : E \to X$ let $\phi(E)$ denote the sheaf of holomorphic sections of the map π. Given a morphism $f : E \to F$ of vector bundles, write $\phi(f) : \phi(E) \to \phi(F)$ for the corresponding morphism of sheaves.

(a) Show that for every E, $\phi(E)$ has a natural structure of an $\mathcal{O}_X$-module.

(b) Show that if E is a holomorphic vector bundle of rank r, then $\phi(E)$ is a coherent sheaf of $\mathcal{O}_X$-modules which is locally free of rank r.

(c) Let $\mathrm{Coh}^{lf}(X)$ be the full subcategory of the category of coherent sheaves on X, consisting of locally free sheaves. Show that $\phi : (\mathsf{Vec/X}) \to \mathrm{Coh}^{\mathsf{lf}}(\mathsf{X})$ is a functor which is an equivalence of categories.

2. Let X be a complex manifold and let $\mathrm{Pic}(X)$ be the set of isomorphism classes of holomorphic line bundles on X.

(a) Show that the tensor product of line bundles turns $\mathrm{Pic}(X)$ into an abelian group.

(b) Show that the functor ϕ transforms the tensor product of vector bundles into the tensor product of $\mathcal{O}_X$-modules. Argue that $\mathrm{Pic}(X)$ can be identified with the group of isomorphism classes of all coherent $\mathcal{O}_X$-modules that are invertible w.r.t. to the tensor product of $\mathcal{O}_X$-modules.

(c) Use the cocycle description of vector bundles to argue that $\mathrm{Pic}(X) = H^1(X, \mathcal{O}^\times)$ as groups.

3. Compute $H^i(\mathbb{P}^1, \mathcal{O}(k))$ for all i and k.

[1] In a different convention one defines the morphisms to be the holomorphic maps of total spaces which are linear and of constant rank on the fibers. We will not use this convention.

4. Let X be a complex manifold and let $\boldsymbol{e} : \mathcal{O}_X \to \mathcal{O}_X^\times$ be the map sending a locally defined holomorphic function f to the locally defined non-vanishing function $\exp(2\pi i f)$.

(a) Show that $\boldsymbol{e}$ is a map of sheaves of abelian groups and that we have a short exact sequence of abelian sheaves

$$0 \to \mathbb{Z}_X \to \mathcal{O}_X \xrightarrow{\boldsymbol{e}} \mathcal{O}_X^\times \to 1.$$

(b) Use **(a)** and the previous problems to compute $\mathrm{Pic}(\mathbb{P}^1)$.