Math 6220, Homework 5, Due Monday, December 11

1. Let $X = \mathbb{P}^2$ with homogeneous coordinates $(x_0 : x_1 : x_2)$. Suppose $p_1, p_2, p_3, p_4 \in X$ are four points in general position, i.e. four distinct points no three of which are collinear. Let V be the vector space of all homogeneous quadratic polynomials $q(x_0, x_1, x_2)$, such that $q(p_i) = 0$ for $i = 1, 2, 3, 4$.

(a) Show that V is two dimensional and that for every point $x \in X - \{p_1, p_2, p_3, p_4\}$ the subspace

$$\ell_x = \{q \in V | q(x) = 0\}$$

is one dimensional.

(b) Consider the map $\varphi : X - \{p_1, p_2, p_3, p_4\} \to \mathbb{P}(V)$, $\varphi(x) = \ell_x$. Let $\widehat{X} \subset X \times \mathbb{P}(V)$ be the closure of the graph of φ. Show that $\widehat{X}$ is isomorphic to the blow-up of X at the four points $p_1, p_2, p_3, p_4 \in X$.

(c) Let $\widehat{\varphi} : \widehat{X} \to \mathbb{P}(V)$ be the natural map. How many critical values does $\widehat{\varphi}$ have?

2. Consider the algebraic curve $X \subset \mathbb{C}^2$ given by the equation $f(x, y) = y^2 - x^4 = 0$. How many times do you need to blow-up $\mathbb{C}^2$ to resolve the singularities of X? Explain your answer.

3. Let $X = \mathbb{P}^2$ with homogeneous coordinates $(x_0 : x_1 : x_2)$. Suppose $p_1, p_2 \in X$ are two distinct points in X. Let W be the vector space of all homogeneous quadratic polynomials $q(x_0, x_1, x_2)$, such that $q(p_i) = 0$ for $i = 1, 2$.

(a) Show that W is 4-dimensional. Show that for every $x \in X - \{p_1, p_2\}$ the subspace

$$\ell_x = \{q \in W | q(x) = 0\}$$

is 3-dimensional.

(b) Consider the map $\varphi : X - \{p_1, p_2\} \to \mathbb{P}(W^\vee)$, $\varphi(x) = [\ell_x]$. Let $\widehat{X} :\subset X \times \mathbb{P}(W^\vee)$ be the closure of the graph of φ. Show that $\widehat{X}$ is isomorphic to the blow-up of X at the four points $p_1, p_2 \in X$.

(c) Show that $\widehat{\varphi}(\widehat{X}) \subset \mathbb{P}(W^\vee)$ is isomorphic to a smooth quadric surface in $\mathbb{P}^3$.

(d) Describe all fibers of the map $\widehat{\varphi} : \widehat{X} \to \mathbb{P}(W^\vee)$.