Math 6220, Homework 4, Due Friday, December 1

1. Let H be the three dimensional complex Heisenberg gorup, i.e. the group

$$H = \left\{ \begin{pmatrix} 1 & z_1 & z_2 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{pmatrix} \middle| \, z_1, z_2, z_3 \in \mathbb{C} \right\}.$$

Describe the Lie algebra $\mathfrak{h}$ of H as an algebra of 3×3 matrices and show that the matrix exponential map $\exp : \mathfrak{h} \to H$ is a biholomorphism.

2. Let $\mathbb{P}^n$ be a complex projective space with homogeneous coordinates $(x_0 : x_1 : \ldots : x_n)$. Let $f_1, \ldots, f_k$ be homogeneous polynomials in the x_i's and let $V = V(f_1, \ldots, f_k)$ be the projective variety they define. Consider the Jacobian matrix $J(f_1, \ldots, f_k)$.

(a) Show that the locus of all points $x \in \mathbb{P}^n$ where $J(f_1, \ldots, f_k)$ has rank $\leq s$ is a projective subvariety. Denote this variety by $j^s V$.

(b) Prove that there exists a minimal integer m such that $j^m V \cap V = V$. If $j^{m-1} V \cap V = \varnothing$ show that V is a compact complex manifold of dimension $n - m$

3. Consider the action of $\mathbb{Z}$ on $\mathbb{C}^2 - \{0\}$ given by $k \cdot (z_1, z_2) := (2^{-k} z_1, 2^{-k} z_2)$ and let $X := (\mathbb{C}^2 - \{0\})/\mathbb{Z}$ be the corresponding Hopf surface. Let $\pi : X \to \mathbb{P}^1$ be the natural holomorphic map with elliptic fibers. Show that if $Z \subset X$ is a compact connected complex submanifold of positive dimension, then either $Z = X$ or Z is equal to a fiber of π.

4. Fix a complex number τ with $\mathrm{Im}(\tau) > 0$. Consider the holomorphic action of the group $G = (\mathbb{C}, +)$ on the complex manifold $\mathbb{C}^\times \times \mathbb{C}^\times$ given by

$$z \cdot (u_1, u_2) := \left(e^{2\pi i z} u_1, e^{2\pi i \tau z} u_2 \right).$$

(a) Show that G acts freely on $\mathbb{C}^\times \times \mathbb{C}^\times$.

(b) Show that the quotient $X := (\mathbb{C}^\times \times \mathbb{C}^\times)/G$ has a natural structure of an elliptic curve, so that the projection map $\mathbb{C}^\times \times \mathbb{C}^\times \to X$ is holomorphic.

(c) Is $\mathbb{C}^\times \times \mathbb{C}^\times \to X$ a holomorphic line bundle?