Math 622, Homework 3, Due Monday, November 6

1. Consider the complex projective line $\mathbb{P}^1$ with its standard structure of a complex manifold.

(a) Show that the real C^∞ manifold underlying $\mathbb{P}^1$ is diffeomorphic to the two sphere S^2.

(b) Let $(\mathbb{R}^3, \langle\ ,\ \rangle)$ be the standard Eucledian 3-space. Let $\{e_1, e_2, e_3\}$ be the standard orthonormal basis and let

$$S^2 := \left\{ (x, y, z) \in \mathbb{R}^3 \,\middle|\, x^2 + y^2 + z^2 = 1 \right\}$$

be the unit sphere.

For any point $p \in S^2$ consider the tangent plane T_pS^2 embedded as an affine plane in $\mathbb{R}^3$ and the operator $J_p : T_pS^2 \to T_pS^2$ of rotation through $\pi/2$ in the positive direction around the normal vector $\overrightarrow{op}$. Explicitly, if we identify T_pS^2 with the vector subspace $V_p := \overrightarrow{op}^\perp$, then for any non-zero vector $\boldsymbol{v} \in V_p$ we define $J_p(\boldsymbol{v})$ as the unique vector in V_p which satisfies

- $\boldsymbol{v} \perp J_p(\boldsymbol{v})$,
- $|\boldsymbol{v}| = |J_p(\boldsymbol{v})|$,
- The bases $\{e_1, e_2, e_3\}$ and $\{\overrightarrow{op}, v, J_p(\boldsymbol{v})\}$ have the same orientation.

Show that the assignment $p \mapsto J_p \in \operatorname{End}(T_pS^2)$ defines an integrable complex structure on S^2.

(c) Show that the complex manifold $\mathbb{P}^1$ is isomorphic to the complex manifold (S^2, J).

2. Let $\mathbb{P}^3$ denote the complex projective 3-space with homogeneous coordinates x_0, x_1, x_2, x_3. Consider the complex submanifold

$$X := \left\{ x \in \mathbb{P}^3 \,\middle|\, x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0 \right\}.$$

Let M be the underlying C^∞ manifold of X and let I denote the corresponding complex structure. Show that (M, I) and $(M, -I)$ are isomorphic as complex manifolds. How can you generalize this example?

3. Let M be a $2n$-dimensional C^∞ manifold. A *Calabi-Yau structure* on M is a complex valued n-form $\sigma \in A^n_{\mathbb{C}}(M) = \Gamma_{C^\infty}(M, \wedge^n(T_M^\vee \otimes \mathbb{C}))$ satisfying

- $d\sigma = 0$;
- $\sigma \wedge \overline{\sigma}$ is everywhere nonzero;
- σ is locally decomposable, i.e. for every point of $p \in M$ we can find a neighborhood $p \in U \subset M$, so that $\sigma_{|U}$ is a product of complex valued one forms.

(a) Consider the map

$$\sigma \wedge \bullet : T_M^\vee \otimes \mathbb{C} \to \wedge^{n+1}(T_M^\vee \otimes \mathbb{C}). \tag{†}$$

Show that the map (†) is a bundle map, i.e. has constant rank at all points $p \in M$.

(b) Define a complex subbundle $E \subset T_M^\vee \otimes \mathbb{C}$ by setting $E := \ker(\sigma \wedge \bullet)$. Show that $\operatorname{rank}_{\mathbb{C}}(E) = n$ and that $T_M^\vee \otimes \mathbb{C} = E \oplus \overline{E}$.

(c) Let $J \in \Gamma_{C^\infty}(M, \operatorname{End}(T_M))$ be the unique almost complex structure for which $E = (T_M^\vee)^{(1,0)}$. Prove that J is integrable and that σ is a holomorphic n-form on (M, J).

4. Let M be a compact 4-dimensional C^∞ manifold and let $\sigma \in A^2(M)$ be a complex valued two form on M satisfying

- $d\sigma = 0$;
- $\sigma \wedge \overline{\sigma}$ is everywhere nonzero;
- $\sigma \wedge \sigma = 0$.

Show that there exists a unique integrable complex structure I on M such that σ is a holomorphic two form on (M, I). Is σ a Calabi-Yau structure on M?