Math 6220, Homework 2, Due Monday, October 3

1. Since $\mathbb{R} \subset \mathbb{C}$, every complex vector space can be viewed as a real vector space. This gives a functor $(\bullet)_{\mathbb{R}} : (\mathsf{Vect}/\mathbb{C}) \to (\mathsf{Vect}/\mathbb{R})$ from the category $(\mathsf{Vect}/\mathbb{C})$ of complex vector spaces to the category $(\mathsf{Vect}/\mathbb{R})$ of real vector spaces. Similarly the process of tensoring a real vector space with $\mathbb{C}$ gives a functor $\bullet \otimes_{\mathbb{R}} \mathbb{C} : (\mathsf{Vect}/\mathbb{R}) \to (\mathsf{Vect}/\mathbb{C})$. Compute the composition functors

$$(\bullet \otimes_{\mathbb{R}} \mathbb{C})_{\mathbb{R}} : (\mathsf{Vect}/\mathbb{R}) \longrightarrow (\mathsf{Vect}/\mathbb{R})$$

$$(\bullet)_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} : (\mathsf{Vect}/\mathbb{C}) \longrightarrow (\mathsf{Vect}/\mathbb{C}).$$

More precisely, check that

(a) $(\bullet \otimes_{\mathbb{R}} \mathbb{C})_{\mathbb{R}}$ is isomorphic to the 'doubling' functor $d_{\mathbb{R}} : (\mathsf{Vect}/\mathbb{R}) \to (\mathsf{Vect}/\mathbb{R})$, $d_{\mathbb{R}}(V) := V \oplus V$, and $d_{\mathbb{R}}(f) := f \oplus f$ for every $f : V \to W$.

(b) $(\bullet)_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to the 'skew-doubling' functor[1] $sd_{\mathbb{C}} : (\mathsf{Vect}/\mathbb{C}) \to (\mathsf{Vect}/\mathbb{C})$, $sd_{\mathbb{C}}(L) := L \oplus \overline{L}$, and $sd_{\mathbb{C}}(f) = f \oplus \bar{f}$ for every $f : L \to M$.

2. Show that the assignment $L \mapsto (L_{\mathbb{R}}, \mathrm{mult}_i)$ gives rise to an equivalence between the category of complex vector spaces and the category of pairs (V, J), where V is a real vector space, $J : V \to V$ is an $\mathbb{R}$-linear operator satisfying $J^2 = -\mathrm{id}$ and a morphism $(V, J) \to (W, K)$ is defined as an $\mathbb{R}$-linear map $f : V \to W$ which intertwines J and K, i.e. for which $K \circ f = f \circ J$.

Suppose that under this equivalence a space L corresponds to a pair (V, J). Compute the pair corresponding to the space $\overline{L}$ in terms of V and J.

[1] Recall that the complex conjugation functor $\overline{(\bullet)} : (\mathsf{Vect}/\mathbb{C}) \to (\mathsf{Vect}/\mathbb{C})$ is defined as follows. Given a complex vector space L one defines the complex-conjugate space $\overline{L}$ as the complex vector space which has the same underlying abelian group as L but is equipped with a new multiplication $*$ by scalars in $\mathbb{C}$, given by $a \star x = \bar{a}x$ for any $a \in \mathbb{C}$ and any $x \in L$. Furthermore for a $\mathbb{C}$-linear map $f : L \to M$ of $\mathbb{C}$-vector spaces, one defines $\bar{f} : \overline{L} \to \overline{M}$ by $\bar{f}(x) := f(x)$ (Check that $\bar{f}$ is a complex linear map!).

3. Let V be a real vector space and let (ω, h, J) be a Kähler structure on V. Show that:

(a) Every two elements in the set $\{\omega, h, J\}$ uniquely determine the third one.

(b) The operator $J : V \to V$ is orthogonal with respect to the metric h.

(c) The decomposition $V \otimes_{\mathbb{R}} \mathbb{C} = V^{1,0} \oplus V^{0,1}$ is orthogonal with respect to h.

(d) The Lefschetz operators $L, \Lambda \in \mathrm{End}(\wedge^\bullet V^\vee)$ are adjoint with respect to the Eucledian metric $\langle \bullet, \bullet \rangle_h$ on $\wedge^\bullet V^\vee$ which is induced from h. That is, show that $\langle L\alpha, \beta \rangle_h = \langle \alpha, \Lambda\beta \rangle_h$ for all $\alpha, \beta \in \wedge^\bullet V^\vee$.

4. Let (V, h) be a Eucledian vector space and suppose I, J, and K are three h-orthogonal complex structures on V that satisfy $IJ = -JI = K$. The datum (h, I, J, K) is called a ***hyper-Kähler structure*** on V.

(a) Show that V becomes in a natural way a module over the algebra $\mathbb{H}$ of quaternions.

(b) Describe the set of all $(a, b, c) \in \mathbb{R}^3$ for which $aI + bJ + cK$ is again an h-orthogonal complex structure on V.

(c) For every $u \in \{I, J, K\}$ consider the associated symplectic form $\omega_u(\bullet, \bullet) := h(u\bullet, \bullet)$. Consider the type decomposition of $\wedge^\bullet(V \otimes \mathbb{C})^\vee$ corresponding to the complex structure I. Show that in this type decomposition the complex 2-form $\omega_J + i\omega_K \in \wedge^2(V \otimes \mathbb{C})^\vee$ has pure type $(2, 0)$ and is non-degenerate on $V^{1,0}$.