Math 6220, Homework 1, Due Wednesday, September 20, 2023

1. Suppose that A is an associative, unital $\mathbb{C}$-algebra of at most countable dimension over $\mathbb{C}$. For any $a \in A$ consider the set of characteristic numbers of a:

$$\mathsf{char}(a) := \{\lambda \in \mathbb{C} | a - \lambda \text{ is not invertible } \}.$$

(i) Show that $\mathsf{char}(a) \neq \varnothing$ for all $a \in A$.

(ii) Show that if A is a division algebra, then $A = \mathbb{C}$.

2. Suppose that A is an associative, unital $\mathbb{C}$-algebra of at most countable dimension over $\mathbb{C}$. Show that an element $a \in A$ is nilpotent if and only if $\mathsf{char}(a) = \{0\}$

3. Let A be a commutative unital $\mathbb{C}$ algebra of finite type. Show that every maximal ideal in A is the kernel of some algebra homomorphism from A to $\mathbb{C}$.

4. Prove Hilbert's Nullstellensatz: For any ideal $\mathfrak{a} \triangleleft \mathbb{C}[x_1, \ldots, x_n]$ we have $I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.