Math 370 Spring 2016
Sample Midterm with Solutions

Contents

1 Problems

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(1) Let A be a 3×3 matrix whose entries are real numbers such that $A^2 = 0$. Show that $I_3 + A$ is invertible.

SOLUTION: 2.1

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(2) Let A and B be symmetric $n \times n$ matrices. Prove that AB is symmetric if and only if $AB = BA$.

SOLUTION: 2.2

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(3) Let $a_1, a_2, \ldots, a_n$ be given real numbers. Calculate

$$\det(A) = \det \begin{bmatrix} a_1 - a_2 & a_2 - a_3 & \ldots & a_{n-1} - a_n & a_n - a_1 \\ a_2 - a_3 & a_3 - a_4 & \ldots & a_n - a_1 & a_1 - a_2 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ a_{n-1} - a_n & a_n - a_1 & \ldots & a_{n-3} - a_{n-2} & a_{n-2} - a_{n-1} \\ a_n - a_1 & a_1 - a_2 & \ldots & a_{n-1} - a_{n-2} & a_{n-1} - a_n \end{bmatrix}$$

SOLUTION: 2.3

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(4) Let G be an abelian group and n a fixed positive integer. Show that the subset of all elements in G whose order divides n is a subgroup in G. Will this be true if G is non-abelian?

SOLUTION: 2.4

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(5) Determine the automorphism group of a cyclic group of order 10.

SOLUTION: 2.5

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(6) Let G be a group and let $x \in G$ be a fixed element. Consider the set $Z = \{y \in G | xy = yx\}$.

(a) Show that Z is a subgroup of G.

(b) Let $H < G$ be the cyclic subgroup generated by x. Show that H is a normal subgroup of Z.

SOLUTION: 2.6

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(7) Consider the subsets H and N of $SL_2(\mathbb{R})$ defined by

$$N = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R}) \middle| ac = 0 \text{ and } bd = 0 \right\}$$

and

$$H = \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \middle| a \neq 0 \right\}$$

Show that N is a subgroup of $SL_2(\mathbb{R})$ and H is a subgroup of H.

SOLUTION: 2.7

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(8) Let G be an abelian group of order 12, and let $\varphi : G \to G$ be the homomorphism given by $\varphi(x) = x^{11}$ for all $x \in G$. Show that φ is an isomorphism.

SOLUTION: 2.8

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(9) The numbers 20604, 53227, 25755, 20927, and 289 are all divisible by 17. Use Cramer's rules to show that

$$\det \begin{bmatrix} 2 & 0 & 6 & 0 & 4 \\ 5 & 3 & 2 & 2 & 7 \\ 2 & 5 & 7 & 5 & 5 \\ 2 & 0 & 9 & 2 & 7 \\ 0 & 0 & 2 & 8 & 9 \end{bmatrix}$$

is also divisible by 17.

Solution: 2.9

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(10) Consider the elements

$$x = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} \qquad y = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}$$

in the symmetric group S_4 of permuations on four letters. Show that x and y are not conjugate in S_4, i.e. show that there is no element $\sigma \in S_4$ satisfying $y = \sigma x \sigma^{-1}$. (Hint: Compute the sign of x and y.)

Solution: 2.10

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(11) Let G be any group. Let $a \in G$ be an element of order 15. Show that there exists an element $x \in G$ such that $x^7 = a$.

Solution: 2.11

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(12) Let $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$, and let $B \in \mathrm{Mat}_{n\times m}(\mathbb{R})$. Suppose $AB = I_m$ and $BA = I_n$. Show that $m = n$.

Solution: 2.12

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2 Solutions

Solution of problem 1: Recall the standard algebraic formula $x^2 - y^2 = (x+y)(x-y)$.

Now thinking of a and b as matrices and sbstituting I_3 for x and A for y we get

$$(I_3 + A)(I_3 - A) = I_3^2 - A^2 = I_3.$$

In other words $I_3 - A$ is the inverse matrix of $I_3 + A$.

Solution of problem 2: AB is symmetric if and only if $AB = (AB)^t = B^t A^t$. Since both A and B are symmetric we have $B^t = B$ and $A^t = A$ and so AB is symmetric if and only if $AB = BA$.

Solution of problem 3: Adding to any row of a matrix a multiple of anothe row does not change the determinant. So the determinant of A is equal to the determinant of the matrix obtained from A by replacing the last row by the sum of all rows. But the sum of all rows has all entries equal to zero and so $\det(A) = 0$.

Solution of problem 4: Let

$$H = \{x \in G \,|\, x \text{ is of order dividing } n\},$$

and let a and b be two elements in H. We need to check that $ab \in H$, and that $a^{-1} \in H$.

To check that $ab \in H$ we need to compute the order of ab. First note that $a, b \in H$ implies $a^n = b^n = e$. Moreover, since G is abelian we have

$$
\begin{aligned}
(ab)^n &= \underbrace{(ab)\cdot(ab)\cdot\dots\cdot(ab)}_{n \text{ times}} \\
&= a^n \cdot b^n \\
&= e \cdot e \\
&= e.
\end{aligned}
$$

Therefore ab has order dividing n.

Similarly

$$(a^{-1})^n \cdot a^n = (a \cdot a^{-1})^n = e^n = e.$$

But $a^n = e$ so we get that $(a^{-1})^n \cdot e = e$, i.e. $(a^{-1})^n = e$. Again this means that the order of a^{-1} divides n.

This reasoning will not work if we can not switch the order of multiplication of a and b. This does not prove that the statement can not be true in a non-abelian group sinc ethre could be some other reasoning that yields the statement. So, to show that the statement does not hold for non-abelian groups we have to exhibit a counter example.

Consider the simplest non-abelian group S_3 and let $H \subset S_3$ be the subset of all elements of order dividing 2. Then

$$
\begin{aligned}
&S_3 = \{\mathbf{1}, (12), (13), (23), (123), (132)\}, \\
&H = \{\mathbf{1}, (12), (13), (23)\}.
\end{aligned}
$$

But $(13)(12) = (123)$ which is of order 3. Hence H is **not** a subgroup.

Solution of problem 5: Let $C_{10} = \{1, x, x^2, \dots, x^9\}$ be a cyclic group of order ten. If $\phi : C_{10} \to C_{10}$ is an automorphism, then ϕ is completely determined by the element $\phi(x)$. The surjectivity of ϕ implies that all elements in C_{10} should be powers of $\phi(x)$ and so $\phi(x)$ must generate C_{10}. In particular this means that $\phi(x)$ has order 10. On the other hand $\phi(x) \in C_{10}$ so we can write $\phi(x) = x^k$ with $0 \leq k \leq 9$. But we know that x^k has order 10 if and only if k and 10 are relatively prime. So we conclude that $1, 3, 7, 9$ are the only possible values of k.

Since ϕ is uniquely determined by $\phi(x)$ we see that $\mathrm{Aut}(C_{10})$ contains exactly four elements ϕ_1, ϕ_3, ϕ_7, and ϕ_9, where

$$\phi_1 : C_{10} \to C_{10}, \quad \phi_1(x^a) = x^a \text{ for all } a,$$

$$\phi_3 : C_{10} \to C_{10}, \quad \phi_3(x^a) = x^{3a} \text{ for all } a,$$

$$\phi_7 : C_{10} \to C_{10}, \quad \phi_7(x^a) = x^{7a} \text{ for all } a,$$

$$\phi_9 : C_{10} \to C_{10}, \quad \phi_9(x^a) = x^{9a} \text{ for all } a.$$

Since ϕ_1 is the identity map, it is the unit element of $\mathrm{Aut}(C_{10})$. Furthermore $\phi_3 \circ \phi_3(x) = \phi_3(\phi_3(x)) = \phi_3(x^3) = x^9$, i.e. $\phi_3 \circ \phi_3 = \phi_9$. Similarly $\phi_3 \circ \phi_3 \circ \phi_3(x) = \phi_3(\phi_9(x)) = \phi_3(x^9) = x^{27} = x^7$, and so $\phi_3 \circ \phi_3 \circ \phi_3 = \phi_7$. Finally $\phi_3 \circ \phi_3 \circ \phi_3 \circ \phi_3(x) = \phi_3(\phi_7(x)) = \phi_3(x^7) = x^{21} = x$, i.e. $\phi_3 \circ \phi_3 \circ \phi_3 \circ \phi_3 = \phi_1$.

This shows that

$$\mathrm{Aut}(C_{10}) = \{\phi_1, \phi_3, (\phi_3)^2, (\phi_3)^3\},$$

i.e. $\mathrm{Aut}(C_{10})$ is a cyclic group of order 4.

Solution of problem 6: If $a, b \in Z$, then $(ab)x = abx = axb = xab = x(ab)$ and so $ab \in Z$. Clearly $1 \cdot x = x = x \cdot 1$ and so $1 \in Z$. Finally if $a \in Z$, then $a^{-1}xa = a^{-1}ax = x$. Multiplying this identity by a^{-1} on the right we get $a^{-1}x = xa^{-1}$ which yields $a^{-1} \in Z$.

Let now $a \in Z$ and let $k \in \mathbb{Z}$. If $k > 0$, then $ax^k = axx^{k-1} = xax^{k-1} = \ldots = x^k a$. Multiplying the last identity by x^{-k} on the left and on the right gives also $ax^{-k} = x^{-k}a$ and so the cyclic subgroup generated by x is normal in Z.

Solution of problem 7: By definition we have

$$N = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R}) \middle| ac = 0 \text{ and } bd = 0 \right\}$$
$$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R}) \middle| \begin{array}{l} \text{either } a = d = 0, c = -b^{-1} \text{ or } b = \\ c = 0, d = a^{-1} \end{array} \right\}$$

In particular $H \subset N$. Now we compute

$$\begin{bmatrix} 0 & b_1 \\ -b_1^{-1} & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & b_2 \\ -b_2^{-1} & 0 \end{bmatrix} = \begin{bmatrix} -b_1/b_2 & 0 \\ 0 & -b_2/b_1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & b \\ -b^{-1} & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -b \\ b^{-1} & 0 \end{bmatrix}$$

$$\begin{bmatrix} a_1 & 0 \\ & a_1^{-1} \end{bmatrix} \cdot \begin{bmatrix} a_2 & 0 \\ & a_2^{-1} \end{bmatrix} = \begin{bmatrix} a_1 a_2 & 0 \\ & a_1^{-1} a_2^{-1} \end{bmatrix}$$

$$\begin{bmatrix} a & 0 \\ & a^{-1} \end{bmatrix}^{-1} = \begin{bmatrix} a^{-1} & 0 \\ & a \end{bmatrix}.$$

This shows that $H < N < SL_2(\mathbb{R})$.

Solution of problem 8: Since G is finite its suffices to check that φ is injective. Write e for the identity element in G. Let x^a and x^b be two elements in G such that $\varphi(x^a) = \varphi(x^b)$. Then $x^{11a} = x^{11b}$ and so the order of x^{a-b} must divide 11. On the other hand x^{a-b} is an element in G and so its order divides $|G| = 12$. Since 11 and 12 are coprime, it follows that the only positive integer that divides the both 11 and 12 is 1, i.e. the order of x^{a-b} is 1 or equivalently $x^{a-b} = e$. This shows that $x^a = x^b$ and so φ is injective.

Solution of problem 9: If $\det A = 0$ then 17 divides $\det A$ and there is nothing to prove. Assume $\det A \neq 0$. We have

$$\begin{bmatrix} 2 & 0 & 6 & 0 & 4 \\ 5 & 3 & 2 & 2 & 7 \\ 2 & 5 & 7 & 5 & 5 \\ 2 & 0 & 9 & 2 & 7 \\ 0 & 0 & 2 & 8 & 9 \end{bmatrix} \cdot \begin{bmatrix} 10^4 \\ 10^3 \\ 10^2 \\ 10 \\ 1 \end{bmatrix} = \begin{bmatrix} 20604 \\ 53227 \\ 25755 \\ 20927 \\ 289 \end{bmatrix}.$$

In other words, the vector

$$X = \begin{bmatrix} 10^4 \\ 10^3 \\ 10^2 \\ 10 \\ 1 \end{bmatrix}$$

is a solution of the linear system

$$AX = B,$$

where

$$A = \begin{bmatrix} 2 & 0 & 6 & 0 & 4 \\ 5 & 3 & 2 & 2 & 7 \\ 2 & 5 & 7 & 5 & 5 \\ 2 & 0 & 9 & 2 & 7 \\ 0 & 0 & 2 & 8 & 9 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 20604 \\ 53227 \\ 25755 \\ 20927 \\ 289 \end{bmatrix}.$$

By Cramer's formulas we have

$$\begin{bmatrix} 10^4 \\ 10^3 \\ 10^2 \\ 10 \\ 1 \end{bmatrix} = \begin{bmatrix} \det A_1/\det A \\ \det A_2/\det A \\ \det A_3/\det A \\ \det A_4/\det A \\ \det A_5/\det A \end{bmatrix},$$

where A_i is the matrix obtained from A by replacing the i-th column by B. But $B = 17B'$, where B' is a vector with integer entries. Thus $\det A_5 = 17 \det A_5'$ where A_5' is the matrix obtained from A by replacing the fifth column by B'. Since all entries of B' are integers, it follows thatall entries of A_5' are integers and so $\det A_5'$ is an integer. From Cramer's formulas we have

$$17 \det A_5' = \det A,$$

nd so 17 divides $\det A$.

Solution of problem 10: If we can find a σ in S_4 satisfying $y = \sigma x \sigma^{-1}$, then $\mathbf{sgn}(y) = \mathbf{sgn}(\sigma x \sigma^{-1}) = \mathbf{sgn}(\sigma)\,\mathbf{sgn}(x)\,\mathbf{sgn}(\sigma)^{-1}$ since

$$\mathbf{sgn} : S_4 \to \{\pm 1\}$$

is a homomorphism. Furthermore, $\{\pm 1\}$ is abelian and so

$$\mathbf{sgn}(\sigma)\,\mathbf{sgn}(x)\,\mathbf{sgn}(\sigma)^{-1} = \mathbf{sgn}(\sigma)\,\mathbf{sgn}(\sigma)^{-1}\,\mathbf{sgn}(x) = \mathbf{sgn}(x).$$

This shows that if x and y are conjugate, then we must have $\mathbf{sgn}(x) = \mathbf{sgn}(y)$. On the other hand, by definition we have $\mathbf{sgn}(x) = \det(P_x)$ and $\mathbf{sgn}(y) = \det(P_y)$, where P_x and P_y are the permutation matrices

$$P_x = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad P_y = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

and so $\det(P_x) = -1$ and $\det(P_y) = 1$, i.e. x and y have different signs. This contradicts the assumption that x and y are conjugate.

Solution of problem 11: The greatest common divisor of 7 and 15 is 1. Therefore we can find integers u and v such that $7u + 15v = 1$. This follows from the classification of subgroups in $\mathbb{Z}$ but in this case we can find u and v explicitly: $7 \cdot (-2) + 15 \cdot 1 = 1$. Now raising a in this power we compute

$$a = a^1 = a^{7\cdot(-2)+15\cdot 1} = \left(a^{-2}\right)^7 \cdot a^{15} = \left(a^{-2}\right)^7 \cdot e = \left(a^{-2}\right)^7.$$

Therefore we can take $x = a^{-2}$.

Solution of problem 12: The rank of A is equal to the number of non-zero rows in the row-reduced echelon form R of A. Since R is obtained from A by left multiplication by elementary matrices, it follows that RB is obtained from AB by left multiplication by elementary matrices. Thus AB and RB are row equivalent. If R has any zero rows at the bottom, then RB will have zero rows at the bottom. Thus $\mathrm{rank}(AB) = \mathrm{rank}(RB) \leq \mathrm{rank}(R) = \mathrm{rank}(A)$. Therefore $\mathrm{rank}(A) \geq m$. Similarly $\mathrm{rank}(BA) \leq \mathrm{rank}(B)$ and hence $\mathrm{rank}(B) \geq n$. On the other hand $\mathrm{rank}(A) \leq \min m, n$ since the rank of A is equal to the number of pivots in the row reduced echelon form of A and by the same reasoning $\mathrm{rank}(B) \leq \min m, n$. This implies $\mathrm{rank}(A) = \mathrm{rank}(B) = m = n$. G
