Math 370, SAMPLE FINAL EXAM

1. Find the number of conjugacy classes in the groups H_8, S_6, and D_4.

2. Let G be a group and suppose $x, y, z \in G$. Prove that xyz, yzx and zxy all have the same order in G. Will xzy have the same order as well?

3. Suppose G is a group and let $N \triangleleft G$ be a finite normal subgroup in G. Show that G must contain a subgroup $H < G$ of finite index, with the property that every element in H commutes with every element in N.

4. Describe the automorphism group $\mathrm{Aut}(S_3)$.

5. Let G be a group with the property that $g^2 = 1$ for all $g \in G$. Show that G is abelian and that G (written additively) can be regarded as a vector space over the field $\mathbb{F}^2$ of two elements.

6. Describe all normal subgroups in A_4. Show by example that $K \triangleleft H$ and $H \triangleleft G$ does not necessarily imply $K \triangleleft G$.

7. Let A be an abelian group, and let A_1, A_2, A_3 be subgroups. Suppose that $\gcd(|A_1|, |A_2|) = 1$, $\gcd(|A_1|, |A_3|) = 1$, and $\gcd(|A_2|, |A_3|) = 1$.

- Show that $A_1A_2A_3$ is a subgroup in A.
- Show that $A_1A_2A_3$ is a direct product. That is, show that $A_1A_2A_3 = A_1 \times A_2 \times A_3$.

8. Let G be a noncommutative group. Prove that $G/Z(G)$ can not be cyclic.

9.

(a) Suppose F is a field, and let V be a vector space over F. Show that the abelian group $(V, +)$ can not be an infinite cyclc group.

(b) Suppose $(V, +)$ is an abelian group. Show that $(V, +)$ can be endowed with the structure of a vector space over $\mathbb{Q}$ if and only if $0 \in V$ is the only element of finite order in the group $(V, +)$, and for every $n \in \mathbb{Z}$ and every $v \in V$ the equation

$$\underbrace{x + \cdots + x}_{n\text{-times}} = v$$

has a solution in V.

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10. Let V be the real vector space consisting of all real valued functions on $\mathbb{R}$. Let $E \subset V$ be the subspace consisting of all even functions and let $O \subset V$ be the subspace consisting of all odd functions. Show that $V = E \oplus O$.

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11. Let F be a finite field with q elements and let V be an n-dimensional vector space over F.

(a) Find the number of bases in V.

(b) Find the number of 2-dimensonal subspaces in V.

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12. Let F be a field. Determine which of the following collections of matrices form a subspace in the vector space $\mathrm{Mat}_{n\times n}(F)$, and compute the dimension of this subspace.

(a) All symmetric matrices.

(b) All matrices with zero determinant.

(c) All matrices with zero trace.

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13. Consider the subspaces $V, W \subset \mathbb{R}^3$ given by

$$V = \mathrm{Span}\left(\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\3\\3\end{bmatrix}\right) \qquad \textit{and} \qquad W = \mathrm{Span}\left(\begin{bmatrix}1\\2\\2\end{bmatrix}, \begin{bmatrix}2\\3\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\-3\end{bmatrix}\right)$$

(a) Find a basis of $V \cap W$.

(b) Find a basis of $V + W$.

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14. Let V be a vector space over a field F and let A and B be subspaces in V. Suppose that $A \cup B = V$. Prove that either $V = A$ or $V = B$.

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15. Let F be a finite field and let $p = \mathrm{char}(F)$.

(a) Prove that F contains $\mathbb{F}_p$ as a subfield.

(b) Prove that addition and multiplication in F endow F with the structure of a vector space over $\mathbb{F}_p$.

(c) Prove that F must have p^n elements for some positive integer n.

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16. Let n be an integer and let F be the subset

$$F = \left\{ \begin{bmatrix} x & y \\ ny & x \end{bmatrix} \;\middle|\; x, y \in \mathbb{F}_5 \right\} \subset \mathrm{Mat}_{2x2}(\mathbb{F}_5)$$

equipped with the usual operations of addition and multiplication of matrices. Find all values of n for which F is a field.
