Math 370. Lecture 10.

Equivalence relations and partitions

Fundamental construction: Start with an algebraic structure and create a new one by equating elements in the underlying set.

To understand how this works we must formalize the notion of "equating" elements. This leads to the concept of partitioning a set.

Def: Let S be a set. A **partition** P of S is a subdivision of S into a union of non-empty, disjoint subsets.

In other words if $\mathcal{P}(S)$ denotes the power set of S (= the set of all possible subsets of S), then

$$P \subset \mathcal{P}(S)$$

is a partition iff:

- $\emptyset \notin P$.
- If $A, B \in P$ then $A \cap B = \emptyset$.
- $S = \bigcup_{A \in P} A$.

Notation: $S = \coprod_{A \in P} A$

will mean that S is a partition of S.

An alternative point of view on partitioning is provided by the notion of an equivalence relation.

Def: Let S be a set. An **equivalence relation** on S is a way "$\sim$" of pairing certain elements of S which is

(transitive) if $a \sim b$ and $b \sim c$, then $a \sim c$.

(symmetric) if $a \sim b$, then $b \sim a$.

(reflexive) $a \sim a$ for all a.

Note: • An equivalence relation $\sim$ is the same thing as a subset

$$R \subset S \times S$$

s.t.

(i) if $(a,b) \in R$, $(b,c) \in R$ then $(a,c) \in R$.

(ii) if $(a,b) \in R$, then $(b,a) \in R$

(iii) $(a,a) \in R \ \forall a$.

- An equivalence relation is the same thing as a partition of S. Indeed:

(a) if "$\sim$" is an equivalence relation on S, then we can partition S into subsets called equivalence classes:

An **equivalence class for "$\sim$"** is a subset of S consisting of all elements that are equivalent to a given element

Since every $a \sim a \Rightarrow$ the equivalence class of a is non-empty and that every a belongs to at least one equivalence class.

Furthermore if C', C'' are two equivalence classes that have a common element, then $C' = C''$.

Indeed, suppose $a \in C' \cap C''$.
If

$$x \in C' \Rightarrow x \sim a$$

and since $a \in C'' \Rightarrow x \in C''$

This shows $C' \subset C''$. Similarly, if $y \in C'' \Rightarrow y \sim a$ and hence $y \in C' \Rightarrow C'' \subset C'$.

This shows that $C' = C''$.

In other words two equivalent classes are either disjoint or coincide. This shows

$$S = \coprod \left(\begin{array}{c} \sim\text{-equivalence} \\ \text{classes} \end{array} \right).$$

<u>Notation:</u> $S/\sim \; \subset \mathcal{P}(S)$ is the partition corresponding to $\sim$, i.e. the set of equivalence classes of $\sim$.

Note that for every partition $\underline{P}$ of S we have a natural map

$$q : S \to \underline{P}$$

$$x \to \bar{x}$$ - the unique set in the partition x belongs to.

So we get a natural map

$$q : S \to S/\sim.$$

(b) Conversely, if $\mathcal{P} \subset \mathcal{P}(S)$ is a partition look at the equivalence relation

$$\text{"}\sim_{\mathcal{P}}\text{"} \text{ on } S$$

given by:

$$x \sim_{\mathcal{P}} y$$

iff x, y belong to the same set in the partition $\mathcal{P}$.

It is automatic to check that "$\sim_{\mathcal{P}}$" is an equivalence relation and that

$$S/\sim_{\mathcal{P}} = \mathcal{P}.$$

Example: Any map of sets

$$e : S \to T$$

defines a partition (and hence an equivalence relation) on S.

Indeed, consider

$$\mathcal{P} = \{ \varphi^{-1}(t) \subset S \mid t \in T \}$$

This is obviously a partition and

- $\mathcal{P} = \varphi(S) \subset T$
- φ factors as

$$\begin{tikzcd} S \arrow[r, "q"] \arrow[rr, bend right, "\varphi"'] & \mathcal{P} \arrow[r, hook, "i"] & T \end{tikzcd}.$$

Suppose now $\varphi : G \to G'$ is a group homomorphism. According to the previous example φ will induce an equivalence relation

$$\text{"}\sim_\varphi\text{" on } G$$

given by

$$a \sim_\varphi b \iff \varphi(a) = \varphi(b).$$

How special is this equivalence relation? The answer is given by the following proposition.

Proposition: If $\varphi : G \to G'$ is a group homomorphism, then $\sim_\varphi$ is uniquely determined by the subgroup

$$N := \ker \varphi .$$

Concretely $a \sim_\varphi b \iff a^{-1}b \in N$.

Proof: Suppose $a \sim_\varphi b$ i.e. $\varphi(a) = \varphi(b)$. Then

$$\varphi(a^{-1}b) = \varphi(a)^{-1}\varphi(b) = e \in G'$$

So $a^{-1}b \in N$.

Conversely if $a^{-1}b \in N$ then

$$\varphi(a^{-1}b) = e \implies \varphi(a)^{-1}\varphi(b) = e$$

$$\implies \varphi(a) = \varphi(b) \qquad \square$$

The equivalence classes for this

equivalence relation are easy to describe in terms of N:

$$\begin{pmatrix}\text{the equivalence} \\ \text{class of } a \in G\end{pmatrix} = \begin{pmatrix} b \in G \text{ s.t. } \\ a \sim_\varphi b \end{pmatrix}$$

$$= \left(b \in G \text{ s.t. } a^{-1} b \in N \right)$$

$$= b \in aN := \{ an \mid n \in N \}$$

So the equivalence classes are all of the form

$$aN$$

for some elements $a \in G$.

<u>Terminology:</u> $aN \subset G$ is called the left coset of N determined by a.

<u>Corollary:</u> A group homomorphism

$$\varphi : G \to G'$$

is injective $\iff$ $\ker \varphi = \{e\}$.

Proof: The fibers of φ are cosets aN. Since each coset aN is bijective to N it follows that φ is injective $\Leftrightarrow$ $N = \{e\}$.

$\square$.

Remark: This gives also an easy necessary condition for a map to be a group homomorphism: If $\varphi : G \to G'$ is a map of groups and φ is a homomorphism, then all fibers of φ over $y \in \varphi(G)$ must be bijective.

Examples:

(1) Consider the homomorphism

$$\mathrm{mult}_n : (\mathbb{Z}, +) \to (\mathbb{Z}, +)$$
$$a \to an$$

$$\mathrm{Ker}(\mathrm{mult}_n) = \{ a \in \mathbb{Z} \mid an = 0 \}$$
$$= \{0\}$$

Thus mult_n is injective.

(2) We can construct an equivalence relation on $\mathbb{Z}$ by the rule:

$$a \sim_n b \iff \left(\begin{array}{l} a-b \text{ is divisible} \\ \text{by } n \end{array} \right)$$

<u>Notation:</u> $a \equiv b \mod n$.

The equivalence classes for this relation are easy to describe:

$$a \equiv b \mod n$$

iff a, b have the same remainder upon division by n.

Therefore

$$\mathbb{Z}/\sim_n = \{ n\mathbb{Z}, 1+n\mathbb{Z}, \dots, (n-1)+n\mathbb{Z} \}$$

i.e. we have exactly n distinct equivalence classes.

(3) Let $\varphi : \mathbb{C}^\times \to \mathbb{R}^\times$ be the absolute value homomorphism. Then the

equivalence classes for $\sim_\varphi$ are the concentric circles centered at the origin.

<u>Note</u>: • $\ker(\varphi) = S^1$

• $r \in \varphi(\mathbb{C}^\times) \iff r > 0$

$$\varphi^{-1}(r) = r S^1$$

the circle of radius r.

(4) Consider

$$\psi : \mathbb{C}^\times \to S^1$$
$$z \to z/|z|$$

This is a group homomorphism. The equivalence classes for $\sim_\psi$ are rays emanating from the origin

(5) Consider

$$p : \mathbb{R}^2 \to \mathbb{R}$$

given by $p\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = y$.

The equivalence classes for $\sim_p$ are lines parallel to the x-axis.

Note that we can define cosets w.r.t. an arbitrary subgroup $H < G$.

Def: The **left cosets** of G w.r.t. $H < G$ are the subsets of the form $aH \subset G$.

The same reasoning as before shows that left cosets partition G and so we get an equivalence relation

$$\sim_H^L \quad \text{on} \quad G.$$

Notation: $G/\sim_H^L$ will be denoted by G/H.

<u>Variant:</u> In the same way we can define the right coset

$$Ha = \{ ha \mid h \in H \}$$

of an element $a \in G$.

The right cosets define an equivalence relation

$$\sim_H^R \text{ on } G$$

and we will write

$$H\backslash G := G/\sim_H^R$$

for the collection of right cosets.

By construction we have quotient maps

$$\begin{cases} q^L : G \to G/H \\ q^R : G \to H\backslash G \end{cases}$$

Def: The number of left cosets of H in G is called the **index** of H in G.

Examples:

(1) Consider the subgroup
$$n\mathbb{Z} < \mathbb{Z}$$
then
$$\mathbb{Z}/n\mathbb{Z} = \{n\mathbb{Z}, 1+n\mathbb{Z}, \dots, 1+(n-1)\mathbb{Z}\}$$
i.e.
$$[\mathbb{Z} : n\mathbb{Z}] = n.$$

(2) $[\mathbb{Z} : \{0\}] = +\infty$

(3) $[\mathbb{C}^\times : S^1] = +\infty$

$[\mathbb{C}^\times : \mathbb{R}^\times_{>0}] = +\infty$

(4) $\left[S^3 : \left\langle \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \right\rangle\right] = 3$

(5) $[\mathbb{H} : \langle i \rangle] = 2.$

Theorem (Lagrange) Let G be a finite group. Let $H < G$, then

$$|G| = |H| \cdot [G : H]$$

Proof: G is a union of $[G:H]$ disjoint coset, each of which is bijective to H $\square$

Corollary: If G is a finite group, $x \in G$ then:

$$|x| \text{ divides } |G|.$$

Proof: $|G| = |\langle x \rangle| \cdot [G : \langle x \rangle]$ $\square$

Corollary: Let G be a finite group and let $|G| = p$ be a prime number.
Let $a \neq e \in G$. Then

$$G = \{1, a, \dots, a^{p-1}\}$$

i.e. G is cyclic of order p.

Proof: $|a|$ divides $|G| = p$.

Since $a \neq e \Rightarrow |a| \neq 1 \Rightarrow |a| = p$.
Thus $|\langle a \rangle| = p \Rightarrow \langle a \rangle = G$

$\square$.

Corollary: Suppose G is finite and

$$\varphi : G \to G'$$

is a homomorphism. Then

$$|G| = |\ker \varphi| \cdot |\operatorname{im} \varphi|$$

Proof: Clear $\square$.

Now we are ready to examine the differences between left and right cosets. Interestingly enough this turns out to be the same as the difference between kernels of homomorphisms and general subgroups.

Examples:

(1) Suppose $\varphi : G \to G'$ is a group homomorphism, and let

$$N := \ker \varphi.$$

Then left cosets for N in G are the same as the right cosets.

Indeed if $a \in G$, $n \in N$, then

$$\begin{aligned} \varphi(an) &= \varphi(a)\varphi(n) \\ &= \varphi(a)\, e \\ &= \varphi(a) \\ &= e\, \varphi(a) \\ &= \varphi(n)\varphi(a) \\ &= \varphi(na). \end{aligned}$$

Thus

$$aN = Na.$$

(2) If $N \triangleleft G$ i.e. if $gNg^{-1} \subset N$ for all $g \in G$ $\Rightarrow$ left cosets for N are the same as right cosets.

Indeed

$$gNg^{-1} \subset N \Rightarrow gN \subset Ng$$

but also

$$gNg^{-1} \subset N \Rightarrow Ng^{-1} \subset g^{-1}N$$

So $aN = Na$ for all a.

(3) It is not necessarily true in general that left and right cosets coincide.

Suppose $G = S_3$

$$H = \left\langle \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \right\rangle .$$

Then the left and right cosets of H in G are different.

(4) If G is abelian and $H < G$ is any group, then $aH = Ha$ for all a since $H \triangleleft G$.

We have the following theorem:

Theorem: Suppose $H < G$ is a subgroup. Then there exists a homomorphism

$$\varphi : G \to G'$$

s.t. $H = \ker \varphi$ iff $H \triangleleft G$.

Proof: If $H = \ker \varphi \Rightarrow H \triangleleft G$.

If $H \triangleleft G$, then G/H inherits a group structure given by the formula

$$(aH) \cdot (bH) = abH$$

and $q : G \to G/H$ is a surjective homomorphism with $\ker q = H$.

$\square$