Solutions to the Midterm Exam, Math 370, Spring 2016

Question 1. Let $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$ and let $B \in \mathrm{Mat}_{m\times k}(\mathbb{R})$. Let $b_1, \ldots, b_k$ be the columns of B and suppose that for every $i = 1, \ldots, k$ the rank of the augmented matrix $[A|b_i]$ is equal to the rank of A. Prove that $\mathrm{rank}[A|B] = \mathrm{rank}(A)$.

Answer 1. The rank r of A is by definition the number of non-zero rows in the row-reduced echelon form R of A. R is obtained by performing a sequence of elementary row operations on A. Performing the same sequence of elementary row operations on $[A|b_i]$ will result in a matrix of the form $[R|b_i']$. If for some zero row of R the corresponding row of $[R|b_i']$ is non-zero, then the row reduced echelon form of $[A|b_i]$ will have at least $r+1$ non-zero rows. But $\mathrm{rank}(A) = \mathrm{rank}([A|b_i]) = r$ so this is impossible. In other words, all entries in b_i' corresponding to the zero rows of R must be equal to zero and hence $[R|b_i']$ is the row reduced echelon form of $[A|b_i]$.

Next perform the same sequence of elementary row operations on the matrix $[A|B]$. This will result in the matrix $[R|B']$, where B' is the $m \times k$ matrix with columns $b_1', \ldots, b_k'$. By the previous reasoning, all the entries in the columns b_i' that correspond to zero rows of R are themselves zero. Therefore a row of R is zero if and only if the corresponding row of $[R|B']$ is zero. This shows that the rank of $[A|B]$ is r. $\square$

Question 2.

(a) Find a polynomial $f(x)$ of degree 3, with real coefficients, and such that $f(-2) = 1$, $f(-1) = 3$, $f(1) = 13$, and $f(2) = 33$.

(b) Prove that there is no polynomial $g(x)$ of degree 2 , with real coefficients, and such that $g(-2) = 1$, $g(-1) = 3$, $g(1) = 13$, and $g(2) = 33$.

Answer 2. If $x_1, \ldots x_n$ are distinct real numbers, and $c_1, \ldots, c_n$ are given real numbers, then we proved in class that there is a unique polynomial $f(x)$ of degree $\leq (n-1)$ satisfying

$f(x_1) = c_1,\ f(x_2) = c_2,\ \ldots,\ f(x_n) = c_n$. Furthermore, by Lagrange interpolation formula this unique polynomial has degree n and is given by

$$f(x) = \sum_{i=1}^{n} c_i \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}.$$

Therefore the answer to (a) is

$$f(x) = 1 \cdot \frac{(x+1)(x-1)(x-2)}{(-2+1)(-2-1)(-2-2)} + 3 \cdot \frac{(x+2)(x-1)(x-2)}{((-1+2)(-1-1)(-1-2)}$$
$$+ 13 \cdot \frac{(x+2)(x+1)(x-2)}{(1+2)(1+1)(1-2)} + 33 \cdot \frac{(x+2)(x+1)(x-1)}{(2+2)(2+1)(2-1)}.$$

For (b) note that there is a unique polynomial g of degree ≤ 3 satisfying the conditions of (b). But f satisfies these conditions and so $g = f$. But f is of degree 3, so there is no g of degree 2. $\square$

Question 3. Let A be an $n \times n$ matrix with real entries.

(a) Let $R_1, \ldots, R_n$ be the rows of A. Suppose that $R_1 + 2R_2 + 3R_3 + \cdots + nR_n$ is the zero row vector. Compute $\det(A)$. Justify your answer.

(b) Suppose that sum of the even numbered columns of A is equal to the sum of the odd numbered columns of A. Compute $\det(A)$. Justify your answer.

Answer 3. (a) Let A' be the matrix obtained from A by the sequence of elementary row operations:

- Replace row_1 by $\text{row}_1 + 2\text{row}_2$;
- Replace row_1 by $\text{row}_1 + 3\text{row}_3$;

$\vdots$

- Replace row_1 by $\text{row}_1 + i\text{row}_i$;

$\vdots$

- Replace row_1 by $\text{row}_1 + n\text{row}_n$;

Then the first row of A' is equal to $R_1 + 2R_2 + \cdots + nR_n = 0$. Therefore $\det(A') = 0$. But these are elementary row operations of type (i) and so do not change the determinant. Hence $\det(A) = \det(A') = 0$.

(b) Since $\det(A) = \det(A^t)$ the question is equivalent to asking for determinant of A^t if we know that the sum of the even numbered rows of A^t is equal to the sum of its odd numbered rows. Let B be the matrix obtained from A^t by adding in sequence row_3, row_5, etc. to row_1, also adding in a sequence row_4, row_6, etc. to row_2. By assumption then the first two rows of B will be equal, and hence $\det(B) = 0$. But B is obtained from A^t by elementary row operations of the first kind and this has the same determinant at A^t. This shows that $\det(A) = \det(A^t) = 0$. $\square$

Question 4. For each of the following statements determine if they are **True or False**. In each case give a reason or a counter example.

(a) The subgroup G of S_8 generated by the transpositions (12), (34), (56), and (78) is commutative.

(b) If $\sigma \in S_{100}$ is an element of order 5, then σ is a cycle.

(c) If $\sigma \in S_4$ is a cycle, then any power of σ is also a cycle.

Answer 4. (a) **(True)** As these are disjoint transpositions they all commute with each other. Also these are transpositions so each of them is equal to its own inverse. Therefore every element in the group they generate is a product of copies of some of these transpositions in some order. Since all terms of such products commute with each other, every two such products will be interchangeable.

(b) **(False)** The order of an element σ in S_{100} is the least common multiple of the lengths of cycles in the disjoint cycle decomposition of σ. So σ is of order 5 if and only if σ decomposes into a product of disjoint cycles of length 5. But there are many disjoint cycles of length 5 in S_{100} and the product of any two of these will be order 5 and will not be a cycle. For instance $\sigma = (1\,2\,3\,4\,5)(6\,7\,8\,9\,10)$ is of order 5 and is not a cycle.
(c) **(False)** For instance $(1\,2\,3\,4)^2 = (1\,3)(2\,4)$. $\square$

Question 5.

(a) Let $\sigma, \tau \in S_n$ be two permutations in n letters. Suppose σ is a cycle of length r. Prove that $\tau\sigma\tau^{-1}$ is also cycle of length r.

(b) Suppose that $\sigma \in S_n$ is an odd permutation. Show that the equation $\sigma\xi = \xi\sigma^2$ can not be solved with any $\xi \in S_n$.

Answer 5. (a) Let $\sigma = (i_1\, i_2\, \ldots\, i_r)$, and let $j \in \{1, \ldots, n\}$. Then

- $\tau\sigma\tau^{-1}(j) = j$ if j is not one of $\tau(i_1), \tau(i_2), \ldots, \tau(i_r))$, and
- $\tau\sigma\tau^{-1}(\tau(i_1)) = \tau(i_2)$, $\tau\sigma\tau^{-1}(\tau(i_2)) = \tau(i_3)$, $\ldots$, $\tau\sigma\tau^{-1}(\tau(i_r)) = \tau(i_1)$.

In other words $\tau\sigma\tau^{-1}$ is the cycle $(\tau(i_1)\, \tau(i_2)\, \ldots\, \tau(i_r))$.

(b) If there is a ξ such that $\sigma\xi = \xi\sigma^2$, then $\mathrm{sgn}(\sigma\xi) = \mathrm{sgn}(\xi\sigma^2)$. Since the sign function is a homomorphism this gives

$$\mathrm{sgn}(\sigma)\mathrm{sgn}(\xi) = \mathrm{sgn}(\xi)\mathrm{sgn}(\sigma)^2.$$

But $\mathrm{sgn}(xi)$ is either $+1$ or -1 so it can be cancelled form both sides. Also $\mathrm{sgn}(\sigma) = -1$ since σ is odd. This gives $-1 = 1$ which is a contradiction. $\square$

Question 6. Let X and Y be two sets and let $f : X \to Y$ be a bijection. Show that the map $\phi : S(X) \to S(Y)$ given by $\phi(\sigma) = f \circ \sigma \circ f^{-1}$ is an isomorphism of symmetric groups.

Answer 6. First we need to check that the map ϕ is well defined. Since the inverse of a bijection is a bijection, and the composition of bijections is a bijection it follows that $f \circ \sigma \circ f^{-1}$ is a bijection from Y to itself. Therefor $\phi(\sigma)$ belongs to $S(Y)$ and ϕ is well defined.

Secondly we must check that ϕ is a homomorphism. Suppose $\sigma, \tau \in S(X)$. Then

$$\begin{aligned}
\phi(\sigma\tau) &= f \circ \sigma\tau \circ f^{-1} \\
&= f \circ \sigma \circ \tau \circ f^{-1} \\
&= f \circ \sigma \circ \mathrm{id}_X \circ \tau \circ f^{-1} \\
&= f \circ \sigma \circ f^{-1} \circ f \circ \tau \circ f^{-1} \\
&= \phi(\sigma) \circ \phi(\tau) \\
&= \phi(\sigma)\phi(\tau).
\end{aligned}$$

Finally we have to check that ϕ is bijective. The map $\psi : S(Y) \to S(X)$ given by $\psi(\alpha) = f^{-1} \circ \alpha \circ f$ obviously satisfies $\phi(\psi(\alpha) = \alpha$ and $\psi(\phi(\sigma)) = \sigma$. Hence ϕ and ψ are inverse functions and so ϕ is a bijection. $\square$
