Solutions to the Midterm Exam, Math 214, Spring 2020

Question 1. True or false. Give a reason or a counter-example

(a) If an $\mathbb{R}$-vector space has a finite generating set, then it is finite dimensional.

(b) A generating subset in a finite dimensional $\mathbb{R}$-vector space must consist of finitely many vectors.

(c) If S is a finite set, and $\mathbb{K}$ is a field, then the vector space $\mathrm{Fun}(S, \mathbb{K})$ of all functions from S to $\mathbb{K}$ is finite dimensional.

Answer 1. Statement (a) is **True** because every finite generating set contains a maximal linearly independent subset and hence contains a basis.

Statement (b) is **False** since the set of all vectors in a vectors space is a spanning set. For instance if we view $V = \mathbb{R}$ as an $\mathbb{R}$-vector space, then V contains infinitely many elements and they trivially genererate V.

Statement (c) is **True** since the collection of delta functions $\{\delta_s\}_{s \in S}$ is a basis of $\mathrm{Fun}(S, \mathbb{K})$. □

Question 2. Let V be a vector space over a field $\mathbb{K}$, and let $\mathbf{x}, \mathbf{y} \in V$ be two vectors, and $a, b \in \mathbb{K}$ be two scalars. Show that

$$a\mathbf{x} + b\mathbf{y} = b\mathbf{x} + a\mathbf{y}$$

if and only if $a = b$ and/or $\mathbf{x} = \mathbf{y}$.

Answer 2. Since

$$a\mathbf{x} + b\mathbf{y} = b\mathbf{x} + a\mathbf{y}$$

the existence of additive inverses for vector addition gives

$$a\mathbf{x} + b\mathbf{y} - b\mathbf{x} - a\mathbf{y} = \mathbf{0}.$$

Commutativity of addition and distributivity of scaling and addition then give

$$(a - b)(\mathbf{x} - \mathbf{y}) = \mathbf{0}.$$

If $a - b \neq 0$ we can multiply both sides of the last identity by $1/(a-b)$ which gives $\mathbf{x} - \mathbf{y} = \mathbf{0}$.

□

Question 3. Which of the following subsets of vectors are vector subspaces. In each case either check the subspace properties or point out a property that fails and explain why.

(a) In the real 2-space $\mathbb{R}^2$ the subset $S \subset \mathbb{R}^2$ of all vectors with integral coordinates:

$$S = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; a, b \in \mathbb{Z} \right\}.$$

(b) in the complex space $\mathbb{C}^\infty$ of all sequences $(a_1, a_2, \ldots, a_n, \ldots)$ of complex numbers (with the term-by-by term addition and scaling) the subset $B \subset \mathbb{C}^\infty$ of all bounded sequences:

$$B = \left\{ (a_i)_{i=1}^\infty \in \mathbb{C}^\infty \;\middle|\; \begin{array}{l} \text{there exists a positive real constant} \\ c > 0 \text{ so that } |a_i| < c \text{ for all } i \end{array} \right\}$$

Answer 3. In part (a) S is not a subspace. It is closed under addition but it is not closed under scaling. Specifically if we scale a vector with integral coordinates by a general real number we will get a vector with non-integral coordinates. For instance

$$\sqrt{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix}.$$

In part (b) S is a subspace. To check this suppose $\mathbf{a} = (a_i)$ and $\mathbf{b} = (b_i)$ are two bounded sequences of complex numbers and α is a real number.

- Since (a_i) is bounded we can find a positive real constant A so that $|a_i| < A$ for all $i = 1, 2, \ldots$. Similarly since (b_i) is bounded we can find a positive real constant B so that $|b_i| < B$ for all $i = 1, 2, \ldots$.

Consider the sum $\mathbf{a}+\mathbf{b}$. Since the sum of sequences is defined term by term it follows that

$$\mathbf{a}+\mathbf{b} = (a_i + b_i)_{i=1}^{\infty}.$$

But by the triangle inequality for the absolute value we have

$$|a_i + b_i| \leq |a_i| + |b_i| < A + B,$$

for all $i = 1, 2, \ldots$. Therefore $\mathbf{a}+\mathbf{b}$ is a bounded sequence as well. This shows that the sum in $\mathbb{R}^{\infty}$ preserves the condition of being bounded.

- Since the scaling of a sequence is defined term by term we have that

$$\alpha\mathbf{a} = (\alpha \cdot a_i)_{i=1}^{\infty}.$$

Then by the multiplicativity of the absolute value we have

$$|\alpha \cdot a_i| = |\alpha| \cdot |a_i| < |\alpha| \cdot A$$

for all $i = 1, 2, \ldots$. This shows that scaling in $\mathbb{R}^{\infty}$ preserves the condition of being bounded.

□

Question 4. Let Pol be the vector space of all polynomials with real coefficients in one variable. Suppose that $V \subset \text{Pol}$ is a vector subspace such that:

- For every $k = 0, 1, 2, \ldots, n$ the subspace V contains a polynomial of degree exactly k. In other words for every $k = 0, 1, 2, \ldots, n$ we have a polynomial $p_k(x) \in V$ such that $p_k(x) = c_k x^k +$ lower degree terms, and $c_k \neq 0$.
- V does not contain any polynomials of degree $> n$.

Show that V nust be equal to the subspace $\text{Pol}_n \subset \text{Pol}$ of polynomials of degree at most n.

Answer 4. By assumption V does not contain any polynomials of degree $> n$. Therefore $V \subset \text{Pol}_n$. To show that $V = \text{Pol}_n$ it suffices to check that V contains a set of polynomials that spans Pol_n.

We are given polynomials $p_0(x)$, $p_1(x)$, ..., $p_n(x)$ in V such that for every $k = 0, 1, \ldots, n$ we have

$$p_k(x) = c_k x^k + \text{ lower degree terms, and } c_k \neq 0.$$

We can use these polynomials to argue that V contains all monomials $1, x, x^2, \ldots, x^n$.

We will argue by induction on n.

Base: $n = 0$. We need to show that $1 \in V$. By assumption we know that we have a polynomial $p_0(x) \in V$ where

$$p_0(x) = c_0, \text{ and } c_0 \neq 0.$$

Since V is a vector subspace we will have that $\frac{1}{c_0}p_0(x) \in V$ But $\frac{1}{c_0}p_0(x) = 1$ hence $1 \in V$.

Step: Suppose that we know that if V contains polynomials $p_0(x), \ldots p_{n-1}(x)$ satisfying

$$p_k(x) = c_k x^k + \text{ lower degree terms, with } c_k \neq 0.$$

for $k = 1, \ldots, n-1$, then V contains the monomials $1, x, \ldots, x^{n-1}$. Suppose in addition V contains a polynomial $p_n(x)$ such that

$$p_n(x) = c_n x^n + \text{ lower degree terms, with } c_n \neq 0.$$

We need to show that V contains the monomial x^n.

Explicitly

$$p_n(x) = c_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} \cdots a_1 x + a_0,$$

and so

$$x^n = \frac{1}{c_n}p_n(x) - \frac{a_{n-1}}{c_n}x^{n-1} - \cdots - \frac{a_1}{c_n}x - \frac{a_0}{c_n}.$$

Since $p_n(x) \in V$ and by the inductive assumption $1, x, \ldots, x^{n-1}$ it follows that the right hand side is a linear combination of polynomials in V. Since V is a vector space this implies $x^n \in V$ and completes the check.

□

Question 5. Let $U \subset \mathrm{Mat}_{2\times 2}(\mathbb{R})$ be the subspace of all symmetric matrices and $V \subset \mathrm{Mat}_{2\times 2}(\mathbb{R})$ be the subspace of all strictly upper triangular matrices:

$$U = \left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} \,\middle|\, a, b, c \in \mathbb{R} \right\},$$

$$V = \left\{ \begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix} \,\middle|\, d \in \mathbb{R} \right\}.$$

(a) Show that $U \oplus V = \mathrm{Mat}_{2\times 2}(\mathbb{R})$.

(b) Decompose the matrix $E = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$ into a sum $E = A + B$ with $A \in U$ and $B \in U$

Answer 5. For part (a) consider the subspace $W := U + V \subset \mathrm{Mat}_{2\times 2}(\mathbb{R})$. Note that $U \cap V = \{\mathbf{0}\}$. Indeed, if $\mathbf{X} \in U \cap V$ is a matrix which is both in U and V, then on one hand we have

$$\mathbf{X} = \begin{pmatrix} a & b \\ b & c \end{pmatrix},$$

and on the other

$$\mathbf{X} = \begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix}.$$

Therefore we must have $b = d$, and $a = 0$, $b = 0$, and $c = 0$. This shows that $W = U \oplus V$. But every matrix in U can be written uniquely as

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} = a \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

is a basis of U and so $\dim U = 3$. Similarly, note that every matrix in V is a scaling of $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and thus $\dim V = 1$. Since $W = U \oplus V$ this imples that $\dim W = \dim U + \dim V = 3 + 1 = 4$. But $\dim \mathrm{Mat}_{2\times 2}(\mathbb{R})$ is also equal to 4 and since W is a subspace we must have $W = \mathrm{Mat}_{2\times 2}(\mathbb{R})$. This proves part (a).

For part (b) we need to solve the equation

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} + \begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix}.$$

This is equivalent to $1 = a$, $1 = b + d$, $2 = b$, and $-1 = c$, and so we get

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}.$$

□

Question 6. Let S be a finite set and let $V = (\mathcal{P}(S), +, \cdot)$ be the power set of S considered as a vector space over $\mathbb{F}_2$ where for $A, B \subset S$, and $\alpha \in \mathbb{F}_2$ we have

$$A + B = A \Delta B = A \cup B - A \cap B$$

$$\alpha \cdot A = \begin{cases} A, & \text{if } \alpha = 1, \\ \varnothing, & \text{if } \alpha = 0, \end{cases}$$

Suppose that X, Y, Z are subsets in S such that $X \not\subset Y \cup Z$, $Y \not\subset X \cup Z$, and $Z \not\subset X \cup Y$. Show that X, Y, and Z are linearly independent when viewed as vectors in V.

Answer 6. We have to check that there is no non-trivial linear combination of X, Y, and Z which is equal to $\mathbf{0} \in V$. Since the coefficients in any linear combination can be equal to either 0 or 1 the non-trivial linear combinations are

$$\begin{aligned}
1 \cdot X + 0 \cdot Y + 0 \cdot Z &= X, \\
0 \cdot X + 1 \cdot Y + 0 \cdot Z &= Y, \\
0 \cdot X + 0 \cdot Y + 1 \cdot Z &= Z, \\
1 \cdot X + 1 \cdot Y + 0 \cdot Z &= X + Y, \\
1 \cdot X + 0 \cdot Y + 1 \cdot Z &= X + Z, \\
0 \cdot X + 1 \cdot Y + 1 \cdot Z &= Y + Z, \\
1 \cdot X + 1 \cdot Y + 1 \cdot Z &= X + Y + Z.
\end{aligned} \tag{1}$$

Since in V the zero vector corresponds to the empty subset $\varnothing \subset S$, we need to show that the subsets in the (1) are never empty.

First note that $\varnothing$ is contained in every subset, and so the conditions $X \not\subset Y \cup Z$, $Y \not\subset X \cup Z$, and $Z \not\subset X \cup Y$ imply that none of X, Y, and Z can be empty.

Let us examine $X + Y$ next. By definition $X + Y = (X \cup Y) - (X \cap Y)$ consists of all points in the unon of X and Y which do not belong simultaneoously in X and Y. But we know that $X \not\subset Y \cup Z$ so we know that there is a point $x \in X$ which does not belong to Y and does not belong to Z. Hence $x \notin X \cap Y$ and so $x \in (X \cup Y) - (X \cap Y)$. This shows that $(X \cup Y) - (X \cap Y)$ is not empty or equivalently that $X + Y \neq \mathbf{0}$. The same reasoning shows that $X + Z \neq \mathbf{0}$ and that $Y + Z \neq \mathbf{0}$.

Finally note that we chose $x \in X$ such that $x \notin Y$ and $x \notin Z$. Thus $x \in X + Y = (X \cup Y) - (X \cap Y)$ but $x \notin (X \cup Y) \cap Z \supset (X + Y) \cap Z$. Therefore $x \in X + Y + Z = ((X + Y) \cup Z) - (X + Y) \cap Z)$. This shows that $X + Y + Z \neq \varnothing$ or equivalently $X + Y + Z \neq \mathbf{0}$. □

Question 7. Let V and W be real vector spaces with bases $\mathbb{E} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\mathbb{F} = \{\mathbf{f}_1, \mathbf{f}_2\}$ respectively. Suppose that the linear map $T : V \to W$ has matrix $\begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 6 \end{pmatrix}$. Find the matrix of T in the bases $\mathbb{E}' = \{\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3\}$ and $\mathbb{F}' = \{\mathbf{f}_1, \mathbf{f}_1 + \mathbf{f}_2\}$.

Answer 7. Write $\mathbf{e}'_1$, $\mathbf{e}'_2$, $\mathbf{e}'_3$ for the elements of the basis $\mathbb{E}'$ and $\mathbf{f}'_1$, $\mathbf{f}'_2$ for the elements of the elements of the basis $\mathbb{F}'$. To compute the matrix of T in these bases we need to compute the coordinates of the vectors in the collection $T(\mathbb{E}')$ in the basis $\mathbb{F}'$. Using the matrix of T in the bases

$\mathbb{E}$ and $\mathbb{F}$ we compute

$$\begin{aligned}
T(\mathbf{e}'_1) &= T(\mathbf{e}_1) = 0 \cdot \mathbf{f}_1 + 3 \cdot \mathbf{f}_2 \\
&= 3\mathbf{f}_2, \\
T(\mathbf{e}'_2) &= T(\mathbf{e}_1 + \mathbf{e}_2) = T(\mathbf{e}_1) + T(\mathbf{e}_2) = (3\mathbf{f}_2) + (1 \cdot \mathbf{f}_1 + 4 \cdot \mathbf{f}_2) \\
&= \mathbf{f}_1 + 7\mathbf{f}_2, \\
T(\mathbf{e}'_3) &= T(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) = T(\mathbf{e}_1) + T(\mathbf{e}_2) + T(\mathbf{e}_3) = \\
&= (3\mathbf{f}_2) + (1 \cdot \mathbf{f}_1 + 4 \cdot \mathbf{f}_2) + (2 \cdot \mathbf{f}_1 + 6 \cdot \mathbf{f}_6) \\
&= 3\mathbf{f}_1 + 13\mathbf{f}_2.
\end{aligned}$$

This gives the vectors $T(\mathbb{E}')$ in terms of the basis $\mathbb{F}$. To get expressions for these vectors in terms of the basis $\mathbb{F}'$ we need to solve for the vectors in $\mathbb{F}$ in terms of the vectors in $\mathbb{F}'$. This is straightforward: since $\mathbf{f}'_1 = \mathbf{f}_1$ and $\mathbf{f}'_2 = \mathbf{f}_1 + \mathbf{f}_2$ we get $\mathbf{f}_1 = \mathbf{f}'_1$ and $\mathbf{f}_2 = -\mathbf{f}'_1 + \mathbf{f}'_2$. Substituting these expression in the previous formulas gives

$$\begin{aligned}
T(\mathbf{e}'_1) &= 3\mathbf{f}_2 = -3\mathbf{f}'_1 + 3\mathbf{f}'_2, \\
T(\mathbf{e}'_2) &= \mathbf{f}_1 + 7\mathbf{f}_2 = -6\mathbf{f}'_1 + 7\mathbf{f}'_2, \\
T(\mathbf{e}'_3) &= 3\mathbf{f}_1 + 13\mathbf{f}_2 = -10\mathbf{f}'_1 + 13\mathbf{f}'_2.
\end{aligned}$$

Hence the matrix of T in the bases $\mathbb{E}'$ and $\mathbb{F}'$ is

$$\begin{pmatrix} -3 & -6 & -10 \\ 3 & 7 & 13 \end{pmatrix}.$$

□

Question 8. Let V be a vector space over a field $\mathbb{K}$ and let $f : V \to \mathbb{K}$ be a linear function which is not identically zero. Consider the subspace $U = \{\mathbf{x} \in V \mid f(\mathbf{x}) = \mathbf{0}\}$ and let $\mathbf{a} \in V$ be any vector that does not belong to U.

(a) Show that for every vector $\mathbf{v} \in V$ the vector

$$\mathbf{x} = \mathbf{v} - \frac{f(\mathbf{v})}{f(\mathbf{a})}\mathbf{a}$$

is well defined and belongs to U.

(b) Show that $U \oplus \operatorname{span}(\mathbf{a}) = V$.

Answer 8. For part (a) note that $\mathbf{a} \notin U$ means $f(\mathbf{a}) \neq 0$ in $\mathbb{K}$. Therefore we can divide by $f(\mathbf{a})$ in $\mathbb{K}$ and so the vector

$$\mathbf{x} = \mathbf{v} - \frac{f(\mathbf{v})}{f(\mathbf{a})}\mathbf{a}$$

is well defined. To check that this vector belongs to U we evaluate f on $\mathbf{x}$:

$$f(\mathbf{x}) = f\left(\mathbf{v} - \frac{f(\mathbf{v})}{f(\mathbf{a})}\mathbf{a}\right) = f(\mathbf{v}) - \frac{f(\mathbf{v})}{f(\mathbf{a})}f(\mathbf{a}) = f(\mathbf{v}) - f(\mathbf{v}) = 0.$$

This shows that $\mathbf{x} \in U$.

For part (b) note that part (a) implies that any vector $\mathbf{v} \in V$ is equal to the sum

$$\mathbf{v} = \mathbf{x} + \frac{f(\mathbf{v})}{f(\mathbf{a})}\mathbf{a},$$

and that $\mathbf{x} \in U$. Since $(f(\mathbf{v})/f(\mathbf{a})) \cdot \mathbf{a}$ is a scaling of $\mathbf{a}$ it belongs to $\operatorname{span}(\mathbf{a})$ and so $V = U + \operatorname{span}(\mathbf{a})$. To check that this is a direct sum we need to check that $U \cap \operatorname{span}(\mathbf{a}) = \{\mathbf{0}\}$.

Suppose $\mathbf{x} \in U \cap \operatorname{span}(\mathbf{a})$. Then $f(\mathbf{x}) = 0$ and $\mathbf{x} = \alpha\mathbf{a}$ for some $\alpha \in \mathbb{K}$. But then $0 = f(\mathbf{x}) = f(\alpha\mathbf{a}) = \alpha f(\mathbf{a})$. Since $f(\mathbf{a}) \neq 0$ it follows that we must have $\alpha = 0$. This implies that $\mathbf{x} = 0 \cdot \mathbf{a} = \mathbf{0}$ and so $U \cap \operatorname{span}(\mathbf{a}) = \{\mathbf{0}\}$.

□
