Math 314 - Lectures 9, 10, and 11

Composition of linear maps

Recall that if X, Y, Z are sets, and

$$f : X \to Y$$

$$g : Y \to Z$$

are maps we can construct a composite map

$$g \circ f : X \to Z$$

defined by

$$g \circ f(x) = g(f(x))$$

for all $x \in X$.

We already used the notion of composition to define invertible (aka bijective) maps of sets.

It turns out that composition preserves the property of maps being linear.

Claim: Let U, V, W be vector spaces over $\mathbb{K}$. Let

$$T : U \to V$$

$$S : V \to W$$

be linear maps. Then the composition map of sets

$$S \circ T : U \to W$$

is also linear.

Proof: Let $x, y \in U$, $c \in \mathbb{K}$. Then

$$\begin{aligned} S \circ T(x+y) &= S(T(x+y)) \\ &= S(T(x) + T(y)) \\ &= S(T(x)) + S(T(y)) \end{aligned}$$

$$= S \circ T(x) + S \circ T(y).$$

Similarly

$$\begin{aligned} S \circ T(cx) &= S(T(cx)) \\ &= S(c\,T(x)) \\ &= c \cdot S(T(x)) \\ &= c \cdot S \circ T(x). \end{aligned}$$

$\square$

Thus the operation of composition gives yet another way of constructing new linear maps out of known ones.

Under our dictionary that uses bases to convert linear maps to matrices and matrices to linear maps, the natural operations on linear maps that produce new linear maps become just the natural operations of matrices.

Theorem: Let U, V, W be $\mathbb{K}$-vector spaces with bases

$$\mathbb{E} = \{e_1, \dots, e_n\} \subset U$$
$$\mathbb{F} = \{f_1, \dots, f_m\} \subset V$$
$$\mathbb{G} = \{g_1, \dots, g_k\} \subset W.$$

Then:

(1) If $T_1, T_2 : U \to V$ are linear maps and $c \in \mathbb{K}$, and if $A_1, A_2 \in \mathrm{Mat}_{m \times n}(\mathbb{K})$ are the matrices of T_1, T_2 in the bases $\mathbb{E}, \mathbb{F}$, then the matrices of

$$T_1 + T_2 : U \to V$$

$$c \cdot T_1 : U \to V$$

in the bases $\mathbb{E}, \mathbb{F}$ are equal to

$$A_1 + A_2 \text{ and } cA_1$$

respectively.

(2) If $T : U \to V$, $S : V \to W$ are linear maps and

- T has a matrix $A \in \mathrm{Mat}_{m \times n}(\mathbb{K})$ in the bases $\mathbb{E}$ and $\mathbb{F}$;

- S has a matrix $B \in \mathrm{Mat}_{k \times m}(\mathbb{K})$ in the bases $\mathbb{F}$ and $\mathbb{G}$.

Then $S \circ T$ has a matrix

$$B \cdot A \in \mathrm{Mat}_{k \times n}(\mathbb{K})$$

in the bases $\mathbb{E}$ and $\mathbb{G}$.

<u>Proof:</u> Follows immediately from the definition of a matrix of a linear map.

For practice let us write down the proof of (2).

Consider the linear map

$$f := S \circ T : V \to W$$

and let C be the matrix of that operator in the

bases $\mathbb{E}$, $\mathbb{G}$.

By definition for every

$$p = 1, \dots, n$$

$$q = 1, \dots, k$$

the entry C_{qp} of C is given by

$$C_{qp} = \left(\begin{array}{l} \text{the } q\text{-th coordinate} \\ \text{of the vector} \\ f(e_p) \text{ in the basis} \\ \{g_1, \dots, g_k\}. \end{array} \right)$$

This can be computed directly:

$$\begin{aligned} f(e_p) &= S \circ T(e_p) \\ &= S(T(e_p)) \\ &= S(A_{1p} f_1 + A_{2p} f_2 + \dots + A_{mp} f_m) \\ &= \sum_{r=1}^{m} A_{rp} S(f_r) \end{aligned}$$

$$= \sum_{r=1}^{m} A_{rp} \left(\sum_{q=1}^{k} B_{qr} g_q \right)$$

$$= \sum_{q=1}^{k} \left(\sum_{r=1}^{m} B_{qr} A_{rp} \right) g_q$$

Hence $$C_{qp} = \sum_{r=1}^{m} B_{qr} A_{rp}$$

$$= \begin{pmatrix} \text{row } q \\ \text{of } B \end{pmatrix} \cdot \begin{pmatrix} \text{column} \\ p \text{ of } A \end{pmatrix}$$

$$= (BA)_{qp}$$

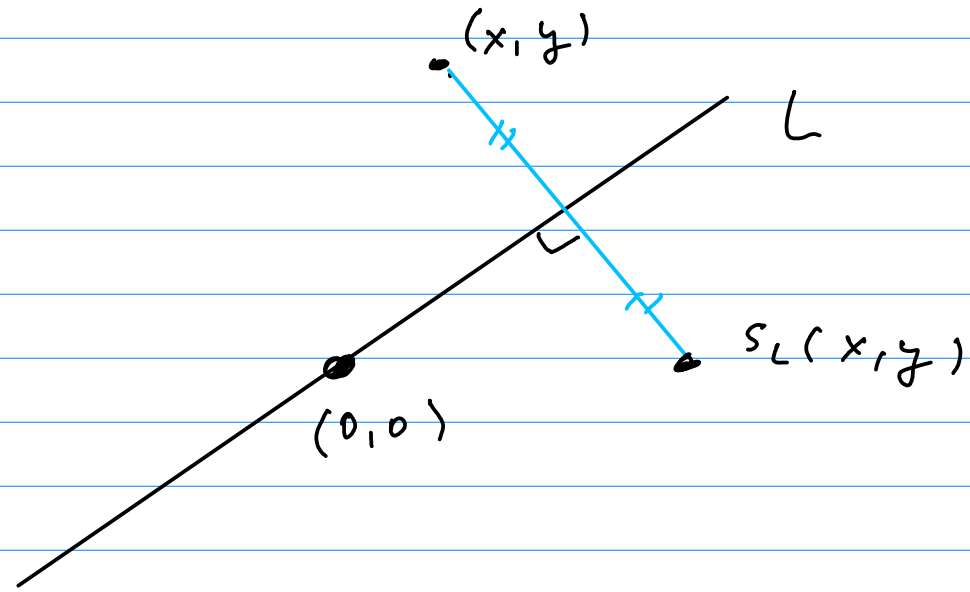
$\square$

Example: (General reflections)

Let $L \subset \mathbb{R}^2$ be any line through the origin and let

$$s_L : \mathbb{R}^2 \to \mathbb{R}^2$$

be the reflection across L. By definition s_L sends a point in $\mathbb{R}^2$ to its mirror image for the mirror L:



Claim: s_L is a linear map.

To check this in general we will need to carry out some complicated geometric constructions.

We can also prove it by realizing that s_L is a composition of linear operations.

Indeed, we already checked that the reflection across the x-axis

$$s = s_{x\text{-axis}} : \mathbb{R}^2 \to \mathbb{R}^2$$

is linear. Indeed

$$s\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x \\ -y \end{pmatrix}$$

for all vectors $\begin{pmatrix} x \\ y \end{pmatrix}$.

This is manifestly linear and in fact in the standard basis of $\mathbb{R}^2$ is given by the matrix

$$A_s := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

But the reflection s_L across L can be viewed as a composition of three transformations:

Step 1: Rotate $\mathbb{R}^2$ so that L becomes the x-axis

Step 2: Reflect $\mathbb{R}^2$ across the x-axis

Step 3: Rotate $\mathbb{R}^2$ so that the x-axis becomes L.

Since each of these steps is linear we conclude that s_L is linear.

Concretely let

$$\gamma = \measuredangle(\text{x-axis}, L)$$

measured in radians and in the counter clockwise direction.

Then

$$s_L = \rho_\gamma \circ s \circ \rho_{-\gamma}$$

In particular we can use this and the previous theorem to compute the matrix of S_L in the standard basis.

We computed

$$A_{S_\gamma} = \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix}$$

$$A_{S_{-\gamma}} = \begin{pmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{pmatrix}$$

$$A_S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

So by the previous theorem we get that

$$A_{S_L} = \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{pmatrix}$$

$$= \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix} \begin{pmatrix} \cos\gamma & \sin\gamma \\ \sin\gamma & -\cos\gamma \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\gamma - \sin^2\gamma & 2\sin\gamma\cos\gamma \\ 2\sin\gamma\cos\gamma & \sin^2\gamma - \cos^2\gamma \end{pmatrix}$$

$$= \begin{pmatrix} \cos 2\gamma & \sin 2\gamma \\ \sin 2\gamma & -\cos 2\gamma \end{pmatrix}$$

<u>Exercise:</u> Compute the matrix of the reflection s_L w.r.t. the line

$$L: \quad y = 7x$$

in the standard basis of $\mathbb{R}^2$.

Properties of composition of linear maps

(1) If $T \in L(V, W)$
$S \in L(U, V)$

then $S \circ T \in L(U, W)$.

(2) Composition is a linear map in each argument.

If $S \in L(U, V)$, then the map

$$S \circ (-) : L(V, W) \to L(U, W)$$

is a linear map between the vector spaces $L(V, W)$ and $L(U, W)$.

Indeed if $a_1, a_2 \in \mathbb{K}$

$T_1, T_2 \in L(V, W)$

Then

$$S \circ (a_1 T_1 + a_2 T_2) = a_1 S \circ T_1 + a_2 S \circ T_2$$

We can check this directly. We just need to show that the values of both sides are equal on any vector

$u \in U$.

From the definition of the $+$ and $\cdot$ in the space of linear maps we compute

$$S \circ (a_1 T_1 + a_2 T_2)(u)$$

$$= S((a_1 T_1 + a_2 T_2)(u))$$

$$= S(a_1 T_1(u) + a_2 T_2(u))$$

$$= a_1 S(T_1(u)) + a_2 S(T_2(u))$$

$$= a_1 (S \circ T)(u) + a_2 (S \circ T_2)(u).$$

which proves the claim.

Similarly if $T \in L(V, W)$ we check directly that

$$(-)\circ T : L(U,V) \to L(U,W)$$

is linear, i.e.

$$(a_1 S_1 + a_2 S_2)\circ T = a_1 S_1 \circ T + a_2 S_2 \circ T$$

for all $a_1, a_2 \in \mathbb{K}$

$S_1, S_2 \in L(U,V)$.

(3) Inverse maps of linear maps are linear.

Suppose that $T: V \to W$ is a linear map which is bijective i.e. is invertible as a map of sets.

Consider the inverse map of sets

$$T^{-1} : W \to V.$$

Then T^{-1} is also linear.

Indeed, let $\vec{y}_1, \vec{y}_2 \in W$

$a_1, a_2 \in \mathbb{K}$.

We need to check that

$$T^{-1}(a_1\vec{y}_1 + a_2\vec{y}_2) = a_1 T^{-1}(y_1) + a_2 T^{-1}(y_2).$$

Let $x_1 = T^{-1}(y_1) \in V$

$x_2 = T^{-1}(y_2) \in V$

Then the RHS is equal to the vector

$$a_1 x_1 + a_2 x_2 \in V.$$

Evaluate T on this vector. Since T is linear we get

$$T(a_1 x_1 + a_2 x_2) = a_1 T(x_1) + a_2 T(x_2)$$

$$= a_1 T \circ T^{-1}(y_1) + a_2 T \circ T^{-1}(y_2)$$

$$= a_1 y_1 + a_2 y_2.$$

Thus

$$T(a_1x_1 + a_2x_2) = a_1y_1 + a_2y_2$$

Now evaluate T^{-1} on both sides of this identity.

We get

$$T^{-1} \circ T(a_1x_1 + a_2x_2) = T^{-1}(a_1y_1 + a_2y_2)$$

$$\parallel$$

$$a_1x_1 + a_2x_2$$

$$\parallel$$

$$a_1T^{-1}(y_1) + a_2T^{-1}(y_2).$$

This proves the linearity of T^{-1}.

Def: A linear map $T: V \to W$ is called an **isomorphism** if there exists a linear map $S: W \to V$ s.t.

$$S \circ T = id_V \quad \text{and} \quad T \circ S = id_W.$$

Note: We checked above that a linear map is an isomorphism if and only if it is a bijection and that the inverse linear map is simply the inverse set theoretic map.

Def: We will say that two vector spaces V, W over $\mathbb{K}$ are **isomorphic** if there exists a linear map

$$T : V \to W$$

which is an isomorphism.

Note: As with sets we will not distinguish isomorphic spaces. The rationale is that using the isomorphism and its inverse we can transport any property or feature of V to W and back. Thus properties and features that only involve the addition and scaling of vectors are matched in

V and W via the isomorphism and its inverse.

Examples:

(1) If V is a finite dimensional vector space over $\mathbb{K}$, then any choice of a basis

$$\mathbb{B} = \{\vec{b}_1, \dots, \vec{b}_n\} \subset V$$

gives an isomorphism

$$[-]_{\mathbb{B}} : V \to \mathbb{K}^n$$

between V and the coordinate n-space $\mathbb{K}^n$.

The map

$$[-]_{\mathbb{B}}$$

assigns to each $\vec{v} \in V$

the column vector of its coordinates in the basis $\mathbb{B}$.

Concretely $\vec{v} \in V$ has a unique representation

$$\vec{v} = \sum_{i=1}^{n} x_i \vec{b}_i$$

with $x_1, \ldots, x_n \in \mathbb{K}$, and

$$[\vec{v}]_{\mathbb{B}} := \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

The inverse map

$$\mathbb{K}^n \to V$$

is also easy to write explicitly (in fact easier):

It is the map that sends a column vector

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{K}^n$$

to the linear combination

$$\sum_{i=1}^{n} x_i \cdot \vec{b}_i \in V.$$

<u>Notation:</u> We can view the linear combination

$$\sum_{i=1}^{n} x_i \vec{b}_i$$

as a formal matrix product. Namely as the product

$$\mathbb{B} \cdot \vec{x}$$

of

- the row vector (of vectors)

$\mathbb{B} = (\vec{b}_1, \ldots, \vec{b}_n)$, and

- the column vector of

numbers

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \mathbb{K}^n.$$

In these terms the inverse map of the 'assign $\mathbb{B}$-coordinates' map

$$[-]_{\mathbb{B}} : V \to \mathbb{K}^n$$

is the 'multiplication by $\mathbb{B}$' map

$$\mathbb{B} \cdot (-) : \mathbb{K}^n \to V$$

<u>Remark:</u> The isomorphism $[-]_{\mathbb{B}}$ between V and $\mathbb{K}$ clearly depends on the choice of the basis $\mathbb{B}$.

Different bases will give rise to different isomorphisms. Another way to describe $[-]_{\mathbb{B}}$ is to say that

$$[-]_{\mathbb{B}} : V \to \mathbb{K}^n$$

is the unique linear map that sends the basis

$$\mathbb{B} = \{ \vec{b}_1, \dots, \vec{b}_n \} \quad \text{of } V$$

to the standard basis

$$\mathbb{E} = \{ \vec{e}_1, \dots, \vec{e}_n \} \quad \text{of } \mathbb{K}^n,$$

i.e. $[-]_{\mathbb{B}}$ is the unique linear map for which

$$[\vec{b}_i]_{\mathbb{B}} = \vec{e}_i \quad \text{for } i = 1, \dots, n$$

or in vector of vectors notation

$$[\mathbb{B}]_{\mathbb{B}} = \mathbb{E}.$$

(2) Let V, W be vector spaces over $\mathbb{K}$ with bases

$$\mathbb{E} = \{ \vec{e}_1, \dots, \vec{e}_n \} \subset V$$

$$\mathbb{F} = \{ \vec{f}_1, \dots, \vec{f}_m \} \subset W$$

respectively.

Given a linear map

$$T : V \to \mathbb{B}$$

let $A_T \in \mathrm{Mat}_{m \times n}(\mathbb{K})$ be the matrix of T in the bases $\mathbb{E}$ and $\mathbb{F}$. The assignment

$$A_{(-)} : L(V, W) \to \mathrm{Mat}_{m \times n}(\mathbb{K})$$
$$T \longrightarrow A_T$$

is a linear map by part (i) of the Theorem on page 4.

This map is an isomorphism. Indeed, by definition A_T is the unique matrix such that

$$T(\mathbb{E}) = \mathbb{F} \cdot A_T .$$

This tells us how to compute the values of T in terms of the

matrix A_T :

If $\vec{v} \in V$ and $\vec{x} \in \mathbb{K}^n$ is the vector of coordinates of $\vec{v}$ in the basis $\mathbb{E}$, then

$$\vec{v} = \mathbb{E}\,\vec{x}$$

and so by the linearity of T we have

$$\begin{aligned} T(\vec{v}) &= T(\mathbb{E}\,\vec{x}) \\ &= T(\mathbb{E})\,\vec{x} \\ &= \mathbb{F}\,A_T\,\vec{x} \end{aligned}$$

Recalling that $\vec{x} = [\vec{v}]_{\mathbb{E}}$ are the coordinates of $\vec{v}$ in the basis $\mathbb{E}$ we get a formula for T in terms of A_T :

$$T(\vec{v}) = \mathbb{F}\,A_T\,[\vec{v}]_{\mathbb{E}}.$$

This gives us the inverse of the map $A_{(-)}$: it is the map

$$\mathrm{Mat}_{m\times n}(\mathbb{K}) \to L(V,W)$$
$$A \to \mathbb{F}A[-]_{\mathbb{E}}.$$

Example (1) above has an important variant.

Claim: Let V, W be vector spaces over $\mathbb{K}$ with bases

$$\mathbb{E} = \{\vec{e}_1, \dots, \vec{e}_n\} \subset V$$
$$\mathbb{F} = \{\vec{f}_1, \dots, \vec{f}_n\} \subset W.$$

Then the unique linear map

$$T : V \to W$$

which sends the basis $\mathbb{E}$ to the basis $\mathbb{F}$ is an isomorphism.

Proof: Let $T : V \to W$ be the unique linear map for which $T(\mathbb{E}) = \mathbb{F}$. Let $S : W \to V$ be the unique linear map for which $S(\mathbb{F}) = \mathbb{E}$.

Note that these maps exist and are unique by the last theorem from lectures 7 and 8.

Then $S \circ T : V \to V$ is a linear map s.t.

$$S \circ T(\mathbb{E}) = S(T(\mathbb{E}))$$
$$= S(\mathbb{F})$$
$$= \mathbb{E}.$$

But we already have a linear map $V \to V$ which sends $\mathbb{E}$ to $\mathbb{E}$, namely id_V.

By the uniqueness part of the last theorem in lecture 7 and 8 it follows that

$$S \circ T = id_V.$$

The same reasoning applied to $T \circ S$ and the basis $\mathbb{F}$ implies that

$$T \circ S = id_W.$$

This completes the proof of the claim.

$\square$

Corollary: Let V, W be finite dimensional vector spaces over $\mathbb{K}$. Then

$$\left(\begin{array}{c} V \text{ and } W \\ \text{are isomorphic} \end{array}\right) \Leftrightarrow \left(\dim V = \dim W\right)$$

Proof: $(\Rightarrow)$ is obvious.

If $T : V \to W$ is an isomorphism, then for any basis $\mathbb{B}$ of V the vectors

$$T(\mathbb{B}) \subset W$$

will be a basis of W.
This can be checked directly.

If $\vec{w} \in W$, then we can consider $\vec{v} = T^{-1}(\vec{w}) \in V$.

By definition $T(\vec{v}) = \vec{w}$.
But

$$\vec{v} \in V = \mathrm{Span}(\mathbb{B})$$

So by the linearity of T we have

$$\vec{w} = T(\vec{v}) \in \mathrm{Span}(T(\mathbb{B})).$$

This shows that $T(\mathbb{B})$ is a generating set of W.

But if $\mathbb{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ and $c_1, \dots, c_n \in \mathbb{K}$ are such that

$$c_1 T(\vec{b}_1) + \dots + c_n T(\vec{b}_n) = \vec{0}$$

in W, then applying T^{-1} to both sides gives

$$\vec{0} = T^{-1}(\vec{0})$$

$$= T^{-1}\left(\sum_{i=1}^{n} c_i\, T(\vec{b}_i)\right)$$

$$= \sum_{i=1}^{n} c_i\, T^{-1}(T(\vec{b}_i))$$

linearity of T^{-1}

$$= \sum_{i=1}^{n} c_i\, \vec{b}_i .$$

Since $\mathbb{B}$ is a basis, this shows that $c_1 = \cdots = c_n = 0$, and hence $T(\mathbb{B})$ is a linearly independent set of W.

So $T(\mathbb{B})$ is a basis and so

$$\dim V = \# \mathbb{B} = \# T(\mathbb{B})$$

$$\| $$

$$\dim W.$$

($\Leftarrow$) Suppose

$$\dim V = \dim W = n$$

and let $\mathbb{E} = \{\vec{e}_1, \dots, \vec{e}_n\} \subset V$

$\mathbb{F} = \{\vec{f}_1, \dots, \vec{f}_n\} \subset W$

be bases.

Then by the previous claim the unique linear map

$$T : V \to W$$

such that $T(\mathbb{E}) = \mathbb{F}$ is an isomorphism.

$\square$.

<u>Notation:</u> If V and W are vector spaces over $\mathbb{K}$ and V and W are isomorphic, then we will write

$$V \cong W.$$

<u>Caution:</u> The previous claim is in its essence an existence statement. We showed that if $\dim V = \dim W$, then there exists a linear map

$$T : V \to W$$

which is an isomorphism. We do not claim here that a randomly chosen linear map will be an isomorphism.

For instance we always have the zero map

$$0 : V \to W$$

which is not an isomorphism since it is constant and hence not bijective (when $\dim V > 0$).

It is therefore useful to have criteria ensuring that a given linear map between spaces of the same dimension is an isomorphism.

One such criterion is given

by the triviality of the kernel of our map.

Def: Let $T : V \to W$ be a linear map between two vector spaces over $\mathbb{K}$. The **kernel** of T is the subspace of V given by

$$\ker T = \{ \vec{v} \in V \mid T(\vec{v}) = \vec{0} \}$$

It turns out that the size of the kernel controls the property of a map being an isomorphism.

Theorem: Let V, W be vector spaces over $\mathbb{K}$ and let

$$T : V \to W$$

be a linear map. Suppose that

$$\dim V = \dim W < \infty$$

Then T is an isomorphism if and only if $\operatorname{Ker} T = \{\vec{0}\}$.

Proof: If T is an isomorphism, then T is bijective and so there is a unique vector

$$\vec{v} \in V \quad \text{such that} \quad T(\vec{v}) = \vec{0}.$$

But T is linear and so $T(\vec{0}) = \vec{0}$. Hence $\vec{0} \in V$ is the unique vector in V that is mapped to $\vec{0}$ by T. This shows that

$$\operatorname{Ker}(T) = \{\vec{0}\}.$$

Conversely, suppose that $\operatorname{Ker}(T) = \{\vec{0}\}$.

Let $\{\vec{e}_1, \dots, \vec{e}_n\}$ be a basis of V.

Then the vectors

$$\vec{f}_1 = T(\vec{e}_1), \dots, \vec{f}_n = T(\vec{e}_n) \in W$$

will be linearly independent. Indeed if we have some scalars $a_1, \ldots, a_n \in \mathbb{K}$ such that

$$a_1 \vec{f_1} + \cdots + a_n \vec{f_n} = \vec{0}$$

in W, then we will have

$$\vec{0} = \sum_{i=1}^{n} a_i \vec{f_i}$$

$$= \sum_{i=1}^{n} a_i T(\vec{e_i})$$

$$= \sum_{i=1}^{n} T(a_i \vec{e_i})$$

$$= T\left(\sum_{i=1}^{n} a_i e_i\right)$$

Since $\ker T = \{\vec{0}\}$ this implies that

$$\sum_{i=1}^{n} a_i \vec{e_i} = \vec{0} \quad \text{in } V.$$

But $\{\vec{e_1}, \ldots, \vec{e_n}\}$ is a basis

of V and so we will have $a_1 = a_2 = \cdots = a_n = 0$.

This shows that $\{\vec{f}_1, \ldots, \vec{f}_n\}$ are linearly independent. Combined with the fact that $\dim W = \dim V = n$ this implies that $\vec{f}_1, \ldots, \vec{f}_n$ is a basis of W.

In other words T sends a basis of V to a basis of W, so T is an isomorphism.

$\square$

We can also detect if a linear map between spaces of the same dimension by looking at the image of that map.

Def: Let $T : V \to W$ be a linear map. The image of T is

$$\operatorname{im} T = \{ \vec{w} \in W \mid \exists \vec{v} \in V \text{ s.t. } T(\vec{v}) = \vec{w} \}$$

Exercise: Check that $\operatorname{im} T$ is always a subspace of W.

Theorem: Let V, W be vector spaces over $\mathbb{K}$ with

$$\dim V = \dim W < \infty.$$

Then a linear map

$$T : V \to W$$

is an isomorphism if and only if $\operatorname{im} T = W$.

Proof: If T is an isomorphism, then T is bijective and in particular surjective. Thus $\operatorname{im} T = W$.

Suppose $\operatorname{im} T = W$ and let $\{\vec{f}_1, \dots, \vec{f}_n\}$ be a basis of W.

Since $\operatorname{im} T = W$ we can find vectors

$$\vec{e}_1, \dots, \vec{e}_n \in V$$

such that $T(\vec{e}_i) = \vec{f}_i$, $i = 1, \dots, n$.

The vectors $\vec{e}_1, \dots, \vec{e}_n$ are linearly independent in V. Indeed, suppose we have

$$a_1, \dots, a_n \in \mathbb{K}$$

so that

$$a_1 \vec{e}_1 + \dots + a_n \vec{e}_n = \vec{0}.$$

Then

$$T\left(\sum_{i=1}^{n} a_i \vec{e}_i\right) = T(\vec{0}) = \vec{0}$$

$$\parallel$$

$$\sum_{i=1}^{n} a_i T(\vec{e}_i)$$

$$\parallel$$

$$\sum_{i=1}^{n} a_i \vec{f}_i.$$

Since $\{\vec{f}_1, \dots, \vec{f}_n\}$ is a basis in V we conclude that $a_1 = \dots = a_n = 0$ which proves the linear independence of $\vec{e}_1, \dots, \vec{e}_n$.

Since $\dim V = \dim W = n$ this

implies that $\{\vec{e_1}, \ldots, \vec{e_n}\}$ is a basis of V.

Again we have shown that T sends a basis to a basis and so T must be an isomorphism.

$\square$

The argument we used to prove the previous two theorems can be refined to show that $\ker T$ and $\operatorname{im} T$ always have complementary dimensions.

Theorem: Let V, W be vector spaces over $\mathbb{K}$, and let

$$T : V \to W$$

be a linear map. Suppose $\dim V < \infty$. Then:

(a) $\dim \operatorname{Ker}(T)$, $\dim \operatorname{Im}(T) < \infty$

(b) $\dim \operatorname{Ker}(T) + \dim \operatorname{Im}(T) = \dim V$.

Proof: For part (a) note that $\operatorname{Ker}(T) \subset V$ is a subspace and by monotonicity of dimension we get

$$\dim \operatorname{Ker}(T) < \dim V < \infty.$$

Also if $\{\vec{e}_1, \dots, \vec{e}_n\}$ is a basis of V then

$$V = \operatorname{span}(\vec{e}_1, \dots, \vec{e}_n)$$

and hence

$$\operatorname{Im}(T) = T(V)$$

$$= \operatorname{span}(T(\vec{e}_1), \dots, T(\vec{e}_n)).$$

Thus $\operatorname{im}(T)$ is spanned by finitely many vectors and so

$$\dim \operatorname{im}(T) < \infty.$$

For part (b) choose a basis

$$\{\vec{e}_1, \dots, \vec{e}_k\} \quad \text{of } \operatorname{Ker}(T)$$

and choose a completion

$$\{\vec{e}_1, \dots, \vec{e}_k, \vec{e}_{k+1}, \dots, \vec{e}_n\}$$

to a basis of V.

Then

$$T(\vec{e}_1), \dots, T(\vec{e}_n)$$

span $\operatorname{im}(T)$.

But $T(\vec{e}_1) = \dots = T(\vec{e}_k) = \vec{0}$ so

$$T(\vec{e}_{k+1}), \dots, T(\vec{e}_n)$$

will span $\operatorname{im}(T)$.

But the vectors

$$T(\vec{e}_{k+1}), \ldots, T(\vec{e}_n)$$

will be linearly independent in $\operatorname{im} T \subset W$.

Indeed if $a_{k+1}, \ldots, a_n \in \mathbb{K}$ are such that

$$\sum_{i=k+1}^{n} a_i T(\vec{e}_i) = \vec{0}$$

then

$$T\left(\sum_{i=k+1}^{n} a_i \vec{e}_i\right) = \vec{0}$$

and so

$$\sum_{i=k+1}^{n} a_i \vec{e}_i \in \operatorname{Ker}(T).$$

So

$$\sum_{i=k+1}^{n} a_i \vec{e}_i = \sum_{j=1}^{k} b_j \vec{e}_j$$

Since $\{\vec{e}_1, \ldots, \vec{e}_n\}$ is a basis of V it follows that

all coefficients must be 0 and so in particular

$$a_{k+1} = \cdots = a_n = 0.$$

Thus $T(\vec{e}_{k+1}), \ldots, T(\vec{e}_n)$ is a basis of $\operatorname{im}(T)$.

This shows that

$$\dim \operatorname{im}(T) = n-k$$

$$= \dim V - \dim \operatorname{Ker}(T).$$

$\square$

Dual spaces and isomorphisms

We can use the above criteria for detecting if a map is an isomorphism to understand better the relationship between a space V and its dual space V^*.

Claim: Let $V/\mathbb{K}$ be a finite dimensional vector space. Then

$$\dim V^\vee = \dim V.$$

Proof: Fix a basis

$$\mathbb{E} = \{e_1, \dots, e_n\} \text{ of } V.$$

Then for every i consider the unique linear function

$$e_i^\vee : V \to \mathbb{K}$$

such that

$$e_i^\vee(e_j) = \begin{cases} 0, & \text{if } j \neq i, \\ 1, & \text{if } j = i. \end{cases}$$

We claim that the functions

$$e_1^\vee, \dots, e_n^\vee \in V^\vee$$

form a basis of the dual space $V^\vee$ of V.

Indeed, let $f \in V^\vee$ be

any element of the dual space. By the properties of linear maps, the linear function

$$f : V \to \mathbb{K}$$

is uniquely determined by its values on the spanning set $\mathbb{E}$.

The values of f on $\mathbb{E}$ are n numbers

$$f(e_1), \ldots, f(e_n) \in \mathbb{K}.$$

Consider the linear function

$$g := f(e_1)\, e_1^\vee + \cdots + f(e_n)\, e_n^\vee,$$

This function has the property that

$$g(e_j) = \sum_{i=1}^{n} f(e_i)\, e_i^\vee(e_j)$$

$$= f(e_j)$$

for all $j = 1, \dots, n$. Thus

$f, g : V \to \mathbb{K}$

take the same values on $e_1, \dots, e_n$ and hence

$$f = g$$

since $\mathbb{E}$ is a spanning set of V.

This shows that

$$V^\vee = \operatorname{span}(e_1^\vee, \dots, e_n^\vee).$$

Next suppose that $c_1, \dots, c_n \in \mathbb{K}$ are such that

$$c_1 e_1^\vee + \dots + c_n e_n^\vee = 0$$

in $V^\vee$.

In other words

$$c_1 e_1^\vee + \dots + c_n e_n^\vee : V \to \mathbb{K}$$

is the zero function.

Evaluating this function on e_j gives

$$c_1 e_1^\vee(e_j) + \cdots + c_n e_n^\vee(e_j) = c_j.$$

But the value of the zero function on any vector is equal to 0. Hence

$$c_1 = \cdots = c_n = 0.$$

Thus $e_1^\vee, \ldots, e_n^\vee$ are linearly independent.

This shows that $e_1^\vee, \ldots, e_n^\vee$ is a basis of $V^\vee$ and hence

$$\dim V^\vee = n = \dim V.$$

$\square$

Def: If V is a vector space over $\mathbb{K}$ with basis $\mathbb{E} = \{e_1, \dots, e_n\}$ we will call the basis of $V^\vee$:

$$\mathbb{E}^\vee = \{e_1^\vee, \dots, e_n^\vee\}$$

the **dual basis** of $\mathbb{E}$.

Note: The previous discussion shows that for any finite dimensional space V with a chosen basis $\mathbb{E}$ we can define an isomorphism

$$T : V \to V^\vee.$$

We define T to be the unique linear map sending the basis $\mathbb{E}$ to the dual basis $\mathbb{E}^\vee$.

This works in the opposite direction as well:

Claim: Suppose V is a finite dimensional vector space over $\mathbb{K}$ and let

$$\mathbb{F} = \{f_1, \dots, f_n\}$$ be

a basis of $V^\vee$. Then there exists a unique basis

$$\mathbb{E} = \{e_1, \dots, e_n\}$$ of V

such that $\mathbb{E}^\vee = \mathbb{F}$.

Proof: Note that each vector $x \in V$ defines a linear function

$$ev_x : V^\vee \to \mathbb{K}$$

$$f \to f(x).$$

This gives a map

$$ev : V \to V^{\vee\vee}$$

$$x \to ev_x.$$

This map is obviously linear: if

$$x_1, x_2 \in V$$

$$a_1, a_2 \in \mathbb{K},$$

then for every $f \in V^\vee$ we have

$$ev_{a_1x_1 + a_2x_2}(f) = f(a_1x_1 + a_2x_2)$$

$$= a_1 f(x_1) + a_2 f(x_2)$$

$$= a_1 ev_{x_1}(f) + a_2 ev_{x_2}(f).$$

By the previous claim

$$\dim V^{\vee\vee} = \dim V^\vee = n$$

so

$$ev : V \to V^{\vee\vee}$$

is a linear map between vector spaces of the same dimension.

Lemma: For a finite dimensional space $V/\mathbb{K}$ the map

$$ev : V \to V^{\vee\vee}$$

is an isomorphism.

Proof: Let $\{b_1, \dots, b_n\}$ be some basis of V, and let

$$\{b_1^\vee, \dots, b_n^\vee\} \subset V^\vee$$

be its dual basis and let

$$\{b_1^{\vee\vee}, \dots, b_n^{\vee\vee}\} \subset V^{\vee\vee}$$

be the basis dual to $\{b_1^\vee, \dots, b_n^\vee\}$.

Let us compute $ev_{b_i}(b_j^\vee)$.

By definition we have

$$ev_{b_i}(b_j^\vee) = b_j^\vee(b_i) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Thus ev_{b_i} and $b_i^{\vee\vee}$ have the same values on

$$\{b_1^\vee, \dots, b_n^\vee\}.$$

Since every linear function is uniquely determined by its values on a spanning set it follows that

$$ev_{b_i} = b_i^{\vee\vee} \quad \text{for } i = 1, \dots, n.$$

Thus ev sends a basis of V to a basis of $V^{\vee\vee}$ and hence ev is an isomorphism.

$\square$

Now let us start with the basis

$$\mathbb{F} = \{f_1, \dots, f_n\} \text{ of } V^\vee.$$

Let

$$\mathbb{F}^\vee = \{f_1^\vee, \dots, f_n^\vee\}$$

be the dual basis of $V^{\vee\vee}$.

Since

$$\ell v : V \to V^{\vee\vee}$$

is an isomorphism, for each $i = 1, \dots, n$ we have a unique vector $e_i \in V$ such that

$$\ell v_{e_i} = f_i^\vee.$$

But then

$$f_i(e_j) = \ell v_{e_j}(f_i) = f_j^\vee(f_i)$$

$$= \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

Consider the collection of vectors

$$\{e_1, \dots, e_n\} \subset V$$

It is a basis of V since it is the image of the basis

$$\{f_1^{\vee}, \dots, f_n^{\vee}\} \text{ of } V^{\vee\vee}$$

under the isomorphism

$$ev^{-1} : V^{\vee\vee} \to V.$$

The formula

$$f_i(e_j) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

implies that

$$\{f_1, \dots, f_n\}$$

is the dual basis of

of $\{e_1, \dots, e_n\}$ and this proves the Claim.

$\square$.

<u>Remark:</u> If V is an n-dimensional space $/\mathbb{K}$, then we saw that

$$V \cong V^\vee$$

and

$$V \cong V^{\vee\vee}.$$

However these two statements are very different in nature:

(1) to construct an isomorphism of V and $V^\vee$ we need a basis of V, while:

(2) there is a canonical isomorphism

$$ev : V \to V^{\vee\vee}$$

which is defined only in terms of V and does not depend on choices.

Exercise: (Duality for linear maps)

Let U, V, W be vector spaces over $\mathbb{K}$.

(1) Show that if

$$T : V \to W$$

is a linear map, then the map

$$T^\vee : W^\vee \to V^\vee$$

defined by

$$T^\vee(f) = f \circ T$$

(for any linear function $f : W \to \mathbb{K}$)

is also linear.

(2) Show that if

$S : U \to V$, $T : V \to W$

are linear maps, then

$$(T \circ S)^\vee = S^\vee \circ T^\vee.$$

(3) Suppose V, W are finite dimensional and let

$$\mathbb{E} = \{e_1, \dots, e_n\} \subset V$$

$$\mathbb{F} = \{f_1, \dots, f_m\} \subset W$$

be bases of V and W.

Let $A \in \mathrm{Mat}_{m \times n}(\mathbb{K})$ be the matrix of T in the bases

$$\mathbb{E} \subset V \quad \text{and} \quad \mathbb{F} \subset W.$$

Show that the matrix of

$$T^\vee : W^\vee \to V^\vee$$

in the dual bases

$$\mathbb{F}^\vee \subset W^\vee \quad \text{and} \quad \mathbb{E}^\vee \subset V^\vee$$

is the matrix

$$A^T \in \mathrm{Mat}_{n \times m}(\mathbb{K}).$$

(4) Prove that if

$$A \in \mathrm{Mat}_{n \times m}(\mathbb{K})$$

$$B \in \mathrm{Mat}_{m \times k}(\mathbb{K})$$

then $(AB)^T = B^T A^T$.