MATH 314 - final exam practice problems, part 2

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11. The numbers 20604, 53227, 25755, 20927 and 289 are all divisible by 17. Show that the following determinant is also divisible by 17.

$$\begin{vmatrix} 2 & 0 & 6 & 0 & 4 \\ 5 & 3 & 2 & 2 & 7 \\ 2 & 5 & 7 & 5 & 5 \\ 2 & 0 & 9 & 2 & 7 \\ 0 & 0 & 2 & 8 & 9 \end{vmatrix}$$

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12. Let $n \geq 2$. Consider the operators

$$\frac{d}{dx}, \frac{d^2}{dx^2} : \mathrm{Pol}_n(\mathbb{R}) \to \mathrm{Pol}_n(\mathbb{R}).$$

True or False. Give a reason or a counter example.

(a) The operators d/dx and d^2/dx^2 have the same invariant subspaces in $\mathrm{Pol}_n(\mathbb{R})$.

(b) The operators d/dx and d^2/dx^2 have the same eigenvectors in $\mathrm{Pol}_n(\mathbb{R})$.

(c) The d/dx and d^2/dx^2 have the same eigenvalues.

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13. Let $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ and $B \in \mathrm{Mat}_{m\times m}(\mathbb{C})$ be fixed complex matrices. Consider the vector space $V = \mathrm{Mat}_{m\times n}(\mathbb{C})$ and the linear operator

$$T : V \to V, \qquad T(X) = BXA.$$

(a) Let $\mathbf{b} \in \mathbb{C}^m$ be an eigenvector of B, and let $\mathbf{a} \in \mathbb{C}^n$ be an eigenvector of A^T. Show that the $m \times n$ matrix $X = \mathbf{b} \cdot \mathbf{a}^T$ is an eigenvector for T.

(b) Suppose A has distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ and B has distinct eigenvalues $\mu_1, \ldots, \mu_m$. Find all eigenvalues of T counting multiplicities.

(c) Suppose A has (not necessarily distinct) eigenvalues $\lambda_1, \ldots, \lambda_n$ counting multiplicities, and suppose B has (not necessarily distinct) eigenvalues $\mu_1, \ldots, \mu_m$ counting multiplicities. Find all eigenvalues of T counting multiplicities.
Hint: Show that you can find sequences of matrices $\{A_k\}_{k=1}^{\infty} \subset \mathrm{Mat}_{n\times n}(\mathbb{C})$, $\{B_\ell\}_{\ell=1}^{\infty} \subset \mathrm{Mat}_{m\times m}(\mathbb{C})$, so that all A_k and B_ℓ have distinct eigenvalues and $\lim_{k\to\infty} A_k = A$, $\lim_{\ell\to\infty} B_\ell = B$. Use this fact together with part (b).

14. Consider the **circulant** matrix

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-3} & a_{n-2} \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ a_2 & a_3 & a_4 & \cdots & a_0 & a_1 \\ a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_0 \end{pmatrix}$$

associated with n numbers $a_0, a_1, \dots, a_{n-1}$.

(a) Let $u_1, u_2, \dots, u_n$ be the n-th roots of unity, that is the n distinct roots of the polynomial $t^n - 1$. Compute the product AW, where W is the Wandermonde matrix

$$W = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ u_1 & u_2 & u_3 & \cdots & u_{n-1} & u_n \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ u_1^{n-2} & u_2^{n-2} & u_3^{n-2} & \cdots & u_{n-1}^{n-2} & u_n^{n-2} \\ u_1^{n-1} & u_2^{n-1} & u_3^{n-1} & \cdots & u_{n-1}^{n-1} & u_n^{n-1} \end{pmatrix}.$$

(b) Use (a) and the multiplicativity of the determinat to show that $\det(A) = f(u_1)f(u_2)\cdots f(u_n)$ where $f(t) = a_0 + a_1 t + a_1 t^2 + \cdots + a_{n-1}t^{n-1}$.

(c) Find the eigenvalues and eigenvectors of A.

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15. True or false. Give a reason or a counter example.

(a) If $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$, and $\mathbf{v}$ is an eigenverctor of A with eigenvalue λ, then $\mathbf{v}$ is an eigenvector of e^A with eigenvalue e^λ.

(b) If $F : V \to V$ is an operator on a finite dimensional complex vector space, then every F-invariant subspace contains an eigenvector for F.

(c) Every permutation matrix in $\mathrm{Mat}_{n\times n}(\mathbb{C})$ is diagonalizable.

(d) If $P \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ is a permutation matrix, then every eigenvalue of P is an eigenvalue of P^{-1}.

(e) If a complex 5×5 matrix A has two distinct eigenvalues, then A must have an eigenvalue with geometric multiplicity 2.

16. Consider the subspaces in $\mathbb{R}^4$:

$$U = \operatorname{span}\left(\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}, \begin{pmatrix}1\\3\\1\\3\end{pmatrix}\right), \qquad \text{and} \qquad V = \operatorname{span}\left(\begin{pmatrix}1\\2\\0\\2\end{pmatrix}, \begin{pmatrix}1\\2\\1\\2\end{pmatrix}, \begin{pmatrix}3\\1\\3\\1\end{pmatrix}\right).$$

Find bases of the subspaces $U + V$ and $U \cap V$ in $\mathbb{R}^4$.

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17. Let V be a finite dimensional real vector space and let $T : V \to V$ be a linear operator.

(a) Suppose that $T - \mathrm{id}_V$ is nilpotent. Show that the operator T is invertible.

(b) Suppose that there existes a polynomial $f(t) \in \mathrm{Pol}(\mathbb{R})$ such that $f(0) \neq 0$ and such that $f(T) = 0$. Show that the operator T is invertible.

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18. Solve the initial value problem

$$\left|\begin{aligned} \frac{dx_1}{dt} &= 3x_1 + 2x_2 - 3x_3, \\ \frac{dx_2}{dt} &= 4x_1 + 10x_2 - 12x_3, \\ \frac{dx_3}{dt} &= 3x_1 + 6x_2 - 7x_3. \end{aligned}\right. \qquad \left|\begin{aligned} x_1(0) &= 1, \\ x_2(0) &= -1, \\ x_3(0) &= 0. \end{aligned}\right.$$

19.

(a) Find the Jordan canonical form of the matrix $J_n(0)^2$.

(b) Classify all nilpotent 5×5 complex matrices A that have a square root.

(c) Classify all nilpotent 6×6 complex matrices that have a square root.

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20. Let V be an n-dimensional space over $\mathbb{C}$ and let $\Gamma \subset L(V, V)$ be a set of commuting operators. Show that there exists a vector $\mathbf{v} \in V$ which is an eigenvector for all $T \in \Gamma$.
