MATH 314 - final exam practice problems, part 1

1. Let X be a finite set with n elements and let $\mathbb{K}$ be a field. Consider the n-dimensional vector space $V = \mathrm{Fun}(X, \mathbb{K})$. Let $0 < m \leq n$ and suppose $U \subset V$ is a subspace of dimension m. Show that there exists a subset $Y \subset X$ of m elements so that the restriction map

$$\begin{aligned} \mathrm{res}: \quad U &\longrightarrow \mathrm{Fun}(Y, \mathbb{K}) \\ f &\longrightarrow f_{|Y} \end{aligned}$$

is an isomorphism of vector spaces.

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2. Let $V = \mathrm{Pol}_n(\mathbb{R})$ be the vector space of all polynomials of degree $\leq n$ with real coefficients. Let $T : V \to V$ be the linear map $T = \mathrm{id}_V - \frac{d^n}{dx^n}$.

(a) Show that T is invertible.

(b) Find the inverse of T as a combination of id_V and the powers of $\frac{d}{dx}$.

(c) Find the Jordan canonical form of T.

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3. Consider the vectors

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{x}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{x}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \qquad \mathbf{x}_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \qquad \text{in } \mathbb{R}^4.$$

Let $V = \mathrm{span}(\mathbf{x}_1, \mathbf{x}_2)$ and $W = \mathrm{span}(\mathbf{x}_3, \mathbf{x}_4)$. Is it true that $\mathbb{R}^4 = V \oplus W$? Justify your answer.

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4. Find the determinant of the $n \times n$ matrix

$$A = \begin{pmatrix} x & y & 0 & \cdots & 0 & 0 \\ 0 & x & y & \cdots & 0 & 0 \\ 0 & 0 & x & \cdots & 0 & 0 \\ & & & \cdots & & \\ 0 & 0 & 0 & \cdots & x & y \\ y & 0 & 0 & \cdots & 0 & x \end{pmatrix}.$$

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5. True or false. Give a reason or a counter example.

(a) A complex $n \times n$ matrix A is invertible if and only the its minimal polynomial has constant term equal to zero.

(b) If $F : V \to V$ is a projector operator on a finite dimensional real vector space, then F has only real eigenvalues.

(c) The determinant of a permutation matrix is always positive.

(d) The product of two permuation matrices is a permutation matrix.

(e) If the Jordan canonical form of a 5×5 complex matrix A has only one Jordan block, then A has an eigenvalue of geometric multiplicity 5.

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6. Cconsider the vectors

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{a}_3 = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}, \qquad \text{and} \qquad \mathbf{b}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}, \quad \mathbf{b}_3 = \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}.$$

Set $U = \operatorname{span}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ and $V = \operatorname{span}(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$. Find bases of the subspaces $U + V$ and $U \cap V$ in $\mathbb{R}^3$.

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7. Show that if $N \in \mathrm{Mat}_{5\times 5}(\mathbb{C})$ is a nilpotent matrix of exponent 5, then there is no 5×5 matrix A satisfying $A^2 = N$.

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8. Compute A^m for the matrix

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}.$$

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9. Let A be a complex $n \times n$ matrix. Show that $\det(e^A) = e^{\mathrm{trace}(A)}$.

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10. Let V be an n-dimensional complex vector space and let $T : V \to V$ be a linear operator of rank one. Show that $T^2 = cT$ for some complex number c.
