
1.

(a) Let $T : V \to W$ be a linear map between finite dimensional vector spaces over a field $\mathbb{K}$. Let $T^\vee : W^\vee \to V^\vee$ be the dual map, i.e. $T^\vee(f) = f \circ T$ for any linear function $f : W \to \mathbb{K}$. Show that the rank of T is equal to the rank of $T^\vee$.

(b) Let $A \in \mathrm{Mat}_{m\times n}(\mathbb{K})$ be an $m \times n$ matrix with entries in $\mathbb{K}$. Show that the rank of A equals the rank of A^T.

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2. Let $T : V \to W$ be a linear map between finite dimensional vector spaces over a field $\mathbb{K}$ and let $T^\vee : W^\vee \to V^\vee$ be its dual.

(a) Show that T is injective if and only if $T^\vee$ is surjective.

(b) Show that T is surjective if and only if $T^\vee$ is injective.

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3. Let $V = \mathrm{Pol}_n$ be the vector space of polynomials of degree $\leq n$ with coefficients in $\mathbb{R}$. For every $a = 0, 1, \ldots, n$ consider the linear function

$$\begin{aligned} \gamma^a : \quad V &\longrightarrow \mathbb{R}, \\ p(x) &\longrightarrow \tfrac{d^a p}{dx^a}(0). \end{aligned}$$

(a) Show that $\{\gamma^0, \gamma^1, \ldots, \gamma^n\}$ is a basis of the dual space $V^\vee$.

(b) Find the unique basis of V for which $\{\gamma^0, \gamma^1, \ldots, \gamma^n\}$ is the dual basis.

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4. Let V be a finite dimensional vector space over $\mathbb{K}$ and let $V^\vee$ be its dual space. Suppose $U \subset V$ is any subset and define a subset $U^\perp \subset V^\vee$ by setting

$$U^\perp = \{\, f \in V^\vee \mid f(x) = 0 \text{ for all } x \in U \,\}.$$

(a) Show that $U^\perp \subset V^\vee$ is always a subspace.

(b) Suppose that $U \subset V$ is a subspace. Show that $\dim U + \dim U^\perp = \dim V$.
