Math 240 Fall 2012
Sample Exam 2 with Solutions

Contents

1 Problems

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(1) Let A be the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Find the sum of the entries in the first row of A^{-1}.

(A) -2
(B) 2
(C) 0
(D) 6
(E) -6
(F) 3

Solution Key: 2.1 Solution: 3.1

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(2) The determinant of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & -1 & 1 & 4 \\ 5 & 6 & 0 & 1 \end{pmatrix}$$

equals

(A) 0

(B) -8

(C) 3

(D) 6

(E) -2

(F) 2

SOLUTION KEY: 2.2 SOLUTION: 3.2

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(3) Which of the following collections of vectors form subspaces in $\mathbb{R}^3$?

(i) $S = \{[a_1, a_2, a_3] \mid a_1, a_2, a_3 \text{ are integers}\}$;

(ii) $S = \{[a_1, a_2, a_3] \mid a_1 + a_2 = 5\}$;

(iii) $S = \{[a_1, a_2, a_3] \mid a_1 + 5a_2 = 0\}$;

(A) (i) and (iii)

(B) (i), (ii), and (iii)

(C) none

(D) (ii)

(E) (iii)

(F) (i) and (ii)

Solution Key: 2.3 Solution: 3.3

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(4) Find all values of λ for which the vectors

$$\begin{pmatrix} 2 \\ -1 \\ \lambda \end{pmatrix}, \qquad \begin{pmatrix} 5 \\ 2\lambda \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ 10 \\ 0 \end{pmatrix}$$

are linearly dependent.

(A) $\lambda = 25, 0$
(B) $\lambda = 4$
(C) $\lambda = \pm 1$
(D) $\lambda = -1, 3, 6$
(E) $\lambda = 8$
(F) all λ

SOLUTION KEY: 2.4 SOLUTION: 3.4

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(5) Let S be the collection of all vectors in $\mathbb{R}^5$ whose first and last coordinate equal each other. S is a subspace in $\mathbb{R}^5$. What is its dimension?

(A) 0

(B) 1

(C) 2

(D) 3

(E) 4

(F) 5

Solution Key: 2.5 Solution: 3.5

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(6) Which of the following collections of matrices form a subspace in the vector space $\mathrm{Mat}_{2\times 2}(\mathbb{R})$?

(a) All symmetric matrices;

(b) All non-singular matrices;

(c) All singular matrices;

(d) All matrices that commute with a fixed matrix B.

(A) only (a)

(B) only (b)

(C) only (c)

(D) only (d)

(E) only (b) and (c)

(F) only (a) and (d)

SOLUTION KEY: 2.6 SOLUTION: 3.6

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(7) True or false. To receive *any* credit, you must also give a reason or counterexample.

(i) The vector space P of all polynomials in x with real coefficients is finite dimensional.

(ii) The functions x, $1+x$ and $\sin(x)$ are linearly independent in the vector space of differentiable functions on the interval $[0,1]$.

(iii) The polynomials $2+x$, x, x^2 span the space P_2 of all polynomials of degree ≤ 2.

(iv) If three vectors v_1, v_2, and v_3 in $\mathbb{R}^3$ span all of $\mathbb{R}^3$, then v_1, v_2, and v_3 are linearly independent.

SOLUTION KEY: 2.7 SOLUTION: 3.7

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(8) Consider the matrix

$$A = \begin{bmatrix} 1 & -3 & 0 & -1 & 2 \\ 3 & -9 & -1 & -6 & 5 \\ 2 & -6 & 0 & -2 & 4 \\ 1 & -3 & 1 & 2 & 3 \end{bmatrix}$$

Which of the following statements is ***false***

(A) $\dim(\text{rowspace}(A)) = 2$

(B) $\dim(\text{rowspace}(A)) + \dim(\text{nullspace}(A)) = 4$

(C) $\dim(\text{rowspace}(A)) + \dim(\text{columnspace}(A)) = 4$

(D) $\dim(\text{columnspace}(A)) + \dim(\text{nullspace}(A)) = 5$

(E) $\text{nullspace}(A) \neq \{\vec{0}\}$

(F) $\dim(\text{columnspace}(A)) = 2$

Solution Key: 2.8 Solution: 3.8

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(9) Let A be the matrix in problem 1.8. Which of the following vectors belongs to the column space of A?

(A) $\begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$ (B) $\begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \end{bmatrix}$ (C) $\begin{bmatrix} -1 \\ -4 \\ 4 \\ 0 \end{bmatrix}$

(D) $\begin{bmatrix} 7 \\ 7 \\ 7 \\ 7 \end{bmatrix}$ (E) $\begin{bmatrix} 0 \\ 1 \\ 7 \\ 0 \end{bmatrix}$ (F) $\begin{bmatrix} 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}$

SOLUTION KEY: 2.9 SOLUTION: 3.9

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(10) Let A be the matrix in problem 1.8. Let

$$\begin{bmatrix} x_1 \\ 0 \\ x_3 \\ 0 \\ 1 \end{bmatrix}$$

be a vector in the null space of A. What is the value of $x_1 + x_3$?

(A) 2
(B) 1
(C) -2
(D) -3
(E) -1
(F) 0

Solution Key: 2.10 Solution: 3.10

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2 Solution key

(1) (C)

(2) (F)

(3) (E)

(4) (A)

(5) (E)

(6) (F)

(7) (i) is false, (ii) is true, (iii) is true, (iv) is true

(8) (B).

(9) (B).

(10) (D).

3 Solutions

Solution of problem 1.1: Using Gaussian elimination we find

$$(A|I) = \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\substack{R_2 \to R_2 - 2R_3 \\ R_1 \to R_1 - 3R_3}}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & -3 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \to R_1 - 2R_2}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right).$$

Hence

$$A^{-1} = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

and the sum of the entries in the first row is 0. The correct answer is (C).

Solution of problem 1.2: We will compute $\det(A)$ by row reduction. We can simplify the calculation by noticing that A is almost upper triangular and so A^T will be almost lower trinagular. This makes makes it easy to compute the row echelon form of A. So

$$\begin{aligned}
\det(A) = \det(A^T) &= \det\begin{pmatrix}1 & 1 & 1 & 5\\ 0 & 2 & -1 & 6\\ 0 & 0 & 1 & 0\\ 0 & 0 & 4 & 1\end{pmatrix}\\
&= 2\det\begin{pmatrix}1 & 1 & 1 & 5\\ 0 & 1 & -\frac{1}{2} & 3\\ 0 & 0 & 1 & 0\\ 0 & 0 & 4 & 1\end{pmatrix} \qquad \left[R_2 \mapsto \frac{1}{2}R_2\right]\\
&= 2\det\begin{pmatrix}1 & 1 & 1 & 5\\ 0 & 1 & -\frac{1}{2} & 3\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \qquad [R_4 \mapsto R_4 - 4R_3]\\
&= 2.
\end{aligned}$$

The correct answer is (F).

Solution of problem 1.3: The collection (i) is ***not*** a subspace. Indeed, the sum of two vectors with integral coordinates will again be a vector with integral coordinates. However if we scale a vector with integral coordinates by an arbitrary real number. The resulting vector will not have integral coordinates. For instance $(1/2) \cdot [1, 0, 0] = [1/2, 0, 0]$ does not have integral coordinates.

The collection (ii) is ***not*** a subspace since it is the set of solutions of an ***inhomogeneous*** linear equation.

The collection (iii) ***is*** a subspace since it is the set of solutions of a ***homogeneous*** linear equation. The correct answer is (E).

Solution of problem 1.4: The vectors will be linearly dependent if and only the matrix

$$A = \begin{pmatrix}2 & -1 & \lambda\\ 5 & 2\lambda & 0\\ 1 & 10 & 0\end{pmatrix}$$

is not of maximal rank, i.e. when $\det(A) = 0$. Expanding this determinant by the last column we get

$$\det(A) = \lambda \cdot (-1)^{1+2} \det \begin{pmatrix} 5 & 2\lambda \\ 1 & 10 \end{pmatrix} = -\lambda(50 - 2\lambda).$$

The vectors will be linearly dependent when $\lambda = 0$ or when $\lambda = 25$. The correct answer is (A).

Solution of problem 1.5: By definition S consists of all vectors $[x_1, x_2, x_3, x_4, x_5]$ such that $x_1 = x_5$. In other words S is the space of solutions of the homogeneous linear system (consistinbg of a single equation):

$$x_1 - x_5 = 0.$$

The coefficient matrix for this system is

$$A = [1, 0, 0, 0, -1].$$

This matrix is already in a row echelon form and has rank 1. Thus the nullspace S of A has dimension

$$\dim S = (\# \text{ of variables}) - \text{rank}(A) = 5 - 1 = 4.$$

The correct answer is (E).

Solution of problem 1.6: The collection (a) is a subspace. Indeed, a matrix A belongs to this collection if and olnly of A satisfies $A^T = A$. By the properties of the transpose we know that $(A+B)^T = A^T + B^T$, and $(cA)^T = cA^T$. In particular, if A and B are symmetric and so satisfy $A^T = A$ and $B^T = B$, we have that $(A+B)^T = A^T + B^T = A + B$, i.e. $A + B$ is also symmetric. Similarly, if A is symmetric and so satisfies $A^T = A$, we have that $(cA)^T = cA^T = cA$, i.e. cA is symmetric as well.

The collection (b) is not a subspace since it does not contain the zero matrix.

The collection (c) is not a subspace since the sum of two singular matrices can be non-singular. Indeed, the diagonal matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

both have determinants equal to zero and are thus singular. On the other hand they add up to the identity matrix which has determinant 1 and is non-singular.

The collection (d) is a subspace. Indeed this collection consists of all 2×2 matrices A that satisfy $AB = BA$. If A_1 and A_2 are two such matrices, then their sum also satifies this condition because of the distributivity of matrix multiplication: $(A_1 + A_2)B = A_1B + A_2B = BA_1 + BA_2 = B(A_1 + A_2)$. Similarly, if A commutes with B, then any scaling of A commutes with B due to the property $c(AB) = (cA)B = A(cB)$.

The correct answer is (F).

Solution of problem 1.7: (i) is false: the infinite collection of all monomials $\{1, x, x^2, \ldots, x^n, \ldots\}$ is linearly indepentent since any finite linear combination of monomials is a polynomial, and a polynomial is the zero polynomial if and only if all of its coefficients are zero. Since P contains infinitely many linearly independent vectors, it can not be finite dimensional.

(ii) is true: The Wronskian for these functions is

$$W(x) = \det \begin{bmatrix} x & 1+x & \sin x \\ 1 & 1 & \cos x \\ 0 & 0 & -\sin x \end{bmatrix} = -x \sin x + (1+x) \sin x = \sin x.$$

Since we can find a point in $[0, 1]$ where the Wronskian does not vanish, e.g. the point $\pi/4$, it follows that the functions are linearly independent.

(iii) is true: If $a_0 + a_1x + a_2x^2$ is any polynomial in P_2, we can always write it as a linear combination of the three given polynomials. To achieve that we have to find constants c_1, c_2, and c_3, so that

$$a_0 + a_1x + a_2x^2 = c_1(2 + x) + c_2x + c_3x^2.$$

Equating the coefficients on both sides we get

$$a_0 = 2c_1, \quad a_1 = c_1 + c_2, \quad a_2 = c_3,$$

and so we get the desired combination by setting $c_1 = a_0/2$, $c_2 = a_1 - a_0/2$, and $c_3 = a_2$.

(iv) is true: Three vectors v_1, v_2, v_3 in $\mathbb{R}^3$ span all of $\mathbb{R}^3$ if and only if the 3×3 matrix A with columns v_1, v_2, v_3 has rank 3. But three vectors in any coordinate space are linearly independent if and only if the matrix with columns those vectors has rank 3. Thus the vectors are linearly independent.

Solution of problem 1.8: Answer (B) is clearly ***false***. The dimensions of the row space and the coulmn space of A are equal and also equal the rank pof A. Therefore answer (B) is equivalent to the statement that $\text{rank}(A) + \text{nullity}(A) = 4$. But by the rank+nullity theorem we have that

$$\text{rank}(A) + \text{nullity}(A) = \# \text{ of columns of } A = 5.$$

The correct answer is (B).

Solution of problem 1.9: To understand the column space of A we need to put A into a reduced echelon form:

$$\begin{bmatrix} 1 & -3 & 0 & -1 & 2 \\ 3 & -9 & -1 & -6 & 5 \\ 2 & -6 & 0 & -2 & 4 \\ 1 & -3 & 1 & 2 & 3 \end{bmatrix} \xrightarrow{\substack{R_2 \mapsto R_2 - 3R_1 \\ R_3 \mapsto R_3 - 2R_1 \\ R_4 \mapsto R_3 - R_1}} \begin{bmatrix} 1 & -3 & 0 & -1 & 2 \\ 0 & 0 & -1 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 1 & -3 & 0 & -1 & 2 \\ 0 & 0 & -1 & -3 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\substack{R_3 \to R_3 - R_2 \\ R_2 \to -R_2}} \begin{bmatrix} 1 & -3 & 0 & -1 & 2 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

A basis for the column space is given by the the columns in A corresponding to the pivot columns of the reduced echelon form. Thus

$$\text{columnspace}(A) = \text{span}\left(\begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right)$$

$$= \left\{ \begin{bmatrix} c_1 \\ 3c_1 - c_2 \\ 2c_1 \\ c_1 + c_2 \end{bmatrix} \;\middle|\; \begin{array}{l} c_1 \text{ and } c_2 \text{ are} \\ \text{any real num-} \\ \text{bers} \end{array} \right\}$$

Therefore the correct answer is (B).

Solution of problem 1.10: From the reduced row echelon form of A we have that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

belongs to the null space of A if and only if

$$x_1 = 3x_2 + x_4 - 2x_5,$$
$$x_3 = -3x_4 - x_5.$$

In particular if $x_2 = 0$, $x_4 = 0$, and $x_5 = 1$ we will have $x_1 = -2$, $x_3 = -1$. This gives $x_1 + x_3 = -3$. The correct answer is (D).