Math 240 Section 002, Fall 2012
Sample Exam 1 with Solutions

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1 Problems

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(1) Let A be a 5×3 matrix of rank two. Which of the following statements are true?

(I) The equation $Ax = 0$ has exactly one solution.

(II) For all vectors b in $\mathbb{R}^5$ the equation $Ax = b$ has infinitely many solutions.

(III) There exists a vector b in $\mathbb{R}^5$ so that the equation $Ax = b$ has a unique solution.

(IV) There exists a vector b in $\mathbb{R}^5$ so that the equation $Ax = b$ has infinitely many solutions.

(A) only (IV) is true.

(B) only (I) and (III) are true.

(C) only (II) and (IV) are true.

(D) only (III) is true.

(E) only (III) and (IV) are true.

(F) only (II) is true.

Solution Key: 2.1 Solution: 3.1

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(2) True or False. Explain your reasoning

(a) The line integral of the vector field $\overrightarrow{F}(x,y) = \langle -2y, x\rangle$ along the counterclockwise oriented boundary of a planar region R is $3 \cdot \text{area}(R)$.

(b) Suppose $\overrightarrow{X} : D \to \mathbb{R}^3$, $(s,t) \mapsto \overrightarrow{X}(s,t)$ is a smooth parametrization of a surface, with D an open subset in the (s,t)-plane. Then for each (s,t) we have

$$\left(\frac{\partial \overrightarrow{X}(s,t)}{\partial s} - 2\frac{\partial \overrightarrow{X}(s,t)}{\partial t}\right) \perp \overrightarrow{N}(s,t).$$

(c) The flux of the vector field $\overrightarrow{F}(x,y,z) = \langle e^{yz}, -y, z\rangle$ through the boundary of solid region R is equal to the volume of R.

Solution Key: 2.2 Solution: 3.2

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(3) Find the surface area of the surface S parametrized by

$$\overrightarrow{X}(s,t) = \left\langle t, s, 5 + \frac{s^2}{2} + \frac{t^2}{2} \right\rangle \quad \text{for} \quad (s,t) \text{ in the disc } D : s^2 + t^2 \leq 1.$$

(A) 1

(B) 8π

(C) $\pi(\sqrt{2} - 1)$

(D) 6

(E) 2

(F) $\dfrac{2\pi}{3}(2^{3/2} - 1)$

Solution Key: 2.3 Solution: 3.3

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(4) In appropriate units the electric charge density $\sigma(x, y, z)$ in a region in space is given by $\sigma = \overrightarrow{\nabla} \cdot \overrightarrow{E} = \mathrm{div}(\overrightarrow{E})$, where $\overrightarrow{E}$ is the electric field. Consider the rectangualr box R given by $0 \leq x \leq 1$, $0 \leq y \leq 2$, $0 \leq z \leq 1$. Compute the total charge

$$\iiint\limits_R \sigma(x, y, z) dx dy dz$$

in this box, if

$$\overrightarrow{E} = \Big\langle (1 - x)\sin(\pi x y z),\ y(2 - y)e^{x^3 + z^3},\ (1 - z)\cos(z) \Big\rangle .$$

(A) 0

(B) -2

(C) -1

(D) 2

(E) 2π

(F) e^2

Solution Key: 2.4 Solution: 3.4

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(5) Use Stokes's theorem to evaluate the line integral $\int_C \overrightarrow{F} \cdot d\overrightarrow{r}$, where

$$\overrightarrow{F}(x, y, z) = \left\langle -y^3, x^3, -z^3 \right\rangle,$$

and C is the curve given by $\overrightarrow{r}(t) = \langle \cos(t),\ \sin(t),\ 1 + \cos(s) \rangle$, $0 \leq t \leq 2\pi$.

(A) $-\pi$

(B) $2\sqrt{2}$

(C) $3 + \pi^3$

(D) $\dfrac{3\pi}{2}$

(E) 1

(F) 0

Solution Key: 2.5 Solution: 3.5

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(6) Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & 3 \\ 1 & 2 & -1 & -1 \\ 0 & 0 & 1 & 4 \end{pmatrix}.$$

Which of the following statements is correct:

(A) The homogeneous system $Ax = 0$ has no solution.

(B) x_2 and x_3 are pivot variables for $Ax = 0$.

(C) If x is a solution of $Ax = 0$, then we must have $x_2 = x_4 = 0$.

(D) $Ax = 0$ has exactly two free variables.

(E) $Ax = 0$ has only one free variable.

(F) $Ax = 0$ has a unique solution.

Solution Key: 2.6 Solution: 3.6

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(7) True or False. Give a reason or an example.

- **(i)** If A is a 5×1 real matrix, and $A^T A = 0$, then A must be the zero matrix.
- **(ii)** There exists 2×3 matrix A such that $A^T A = AA^T$.
- **(iii)** Let A, B be $n \times n$ matrices with A - invertible. Then $(A^{-1}BA)^2 = A^{-1}B^2A$.

Solution Key: 2.7 Solution: 3.7

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(8) Let S be the lateral surface of the cylinder $x^2 + y^2 = 3$, $0 \leq z \leq 4$ oriented by the outward pointing normal. Evaluate $\iint\limits_S \overrightarrow{F} \cdot d\overrightarrow{S}$ for $\overrightarrow{F}(x, y, z) = \langle 2x, 2y, z \ln(1 + x^2) \rangle$.

(a) 48π

(b) 2

(c) 2π

(d) 0

(e) 16π

(f) 1

Solution Key: 2.8 Solution: 3.8

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(9) Let S be the surface given by $\overrightarrow{X}(s,t) = se^t\,\widehat{\boldsymbol{\imath}} + (s+t)\,\widehat{\boldsymbol{\jmath}} + e^{st}\,\widehat{\boldsymbol{k}}$. What is the unit normal vector to S at the point $(0,0,1)$?

(a) $\widehat{\boldsymbol{k}}$

(b) $e\widehat{\boldsymbol{\imath}} + 2\widehat{\boldsymbol{\jmath}}$

(c) $\widehat{\boldsymbol{\imath}} - 2\widehat{\boldsymbol{\jmath}} + \widehat{\boldsymbol{k}}$

(d) $\widehat{\boldsymbol{\jmath}} + \widehat{\boldsymbol{k}}$

(e) $\widehat{\boldsymbol{\jmath}}$

(f) $\dfrac{1}{\sqrt{2}}\widehat{\boldsymbol{\imath}} + \dfrac{1}{\sqrt{2}}\widehat{\boldsymbol{\jmath}}$

Solution Key: 2.9 Solution: 3.9

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(10) How many 2×2 matrices X will solve the matrix equation

$$\begin{pmatrix} 4 & 6 \\ 6 & 9 \end{pmatrix} X = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

(a) none

(b) one

(c) two

(d) three

(e) four

(f) infintely many

Solution Key: 2.10 Solution: 3.10

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2 Solution key

(1) (A)

(2) (a) is True, (b) is True, (c) is False

(3) (F)

(4) (B)

(5) (D)

(6) (D)

(7) (i) is True, (ii) is False, (iii) is True

(8) (A)

(9) (A)

(10) (A).

3 Solutions

Solution of problem 1.1: Since the rank of A is two and the number of variables is 3 the homogeneous system $Ax = 0$ must have infinitely many solutions. Thus (I) is false and (IV) is true. If the vector b in $\mathbf{R}^5$ is such that $Ax = b$ has a solution say x_o, then every solution of $Ax = b$ will be of the form $x = x_o + x_h$ where x_h is some solution of the homogeneous system $Ax = 0$. But we already saw that $Ax = 0$ has infinitely many solutions and so $Ax = b$ has either no solutions or infinitely many solutions. Therefore (II) and (III) are both false. The correct answer is (A).

Solution of problem 1.2: (a) By Green's theorem

$$\begin{aligned}\int_{\partial R} \overrightarrow{F} \cdot d\overrightarrow{r} &= \iint\limits_R \left(\frac{\partial x}{\partial x} - \frac{\partial(-2y)}{\partial y} \right) dxdy \\ &= \iint\limits_R (1+2)dxdy = 3 \iint\limits_R dxdy \\ &= 3\text{area}(R).\end{aligned}$$

So (a) is True.

(b) By definition

$$\overrightarrow{N}(s,t) = \frac{\partial \overrightarrow{X}(s,t)}{\partial s} \times \frac{\partial \overrightarrow{X}(s,t)}{\partial t},$$

and so $\overrightarrow{N}(s,t)$ is perpendicular to both $\partial\overrightarrow{X}/\partial s$ and $\partial\overrightarrow{X}/\partial t$. So $\overrightarrow{N} \cdot \partial\overrightarrow{X}/\partial s = 0$ and $\overrightarrow{N} \cdot \partial\overrightarrow{X}/\partial t = 0$. But then

$$\overrightarrow{N} \cdot \left(\frac{\partial \overrightarrow{X}}{\partial s} - 2\frac{\partial \overrightarrow{X}}{\partial t} \right) = 0 - 2 \cdot 0 = 0,$$

and hence

$$\overrightarrow{N} \perp \left(\frac{\partial \overrightarrow{X}}{\partial s} - 2\frac{\partial \overrightarrow{X}}{\partial t} \right).$$

So (b) is True.

(c) By Gauss's theorem we can compute the flux:

$$\begin{aligned}\iint\limits_{\partial R} \overrightarrow{F} \cdot d\overrightarrow{S} &= \iiint\limits_{R} \overrightarrow{\nabla} \cdot \overrightarrow{F} dV \\ &= \iiint\limits_{R} (0 - 1 + 1) dV \\ &= 0.\end{aligned}$$

So (c) is False.

Solution of problem 1.3: The normal vector to this surface is

$$\begin{aligned}\overrightarrow{N}(s,t) &= \frac{\partial \overrightarrow{X}(s,t)}{\partial s} \times \frac{\partial \overrightarrow{X}(s,t)}{\partial t} \\ &= \langle 0, 1, s\rangle \times \langle 1, 0, t\rangle \\ &= \det \begin{pmatrix} \widehat{\boldsymbol{\imath}} & \widehat{\boldsymbol{\jmath}} & \widehat{\boldsymbol{k}} \\ 0 & 1 & s \\ 1 & 0 & t \end{pmatrix} \\ &= t\widehat{\boldsymbol{\imath}} + s\widehat{\boldsymbol{\jmath}} - \widehat{\boldsymbol{k}}.\end{aligned}$$

This gives $|\overrightarrow{N}| = \sqrt{1 + s^2 + t^2}$ and therefore

$$\text{area}(S) = \iint\limits_{S} dS = \iint\limits_{s^2+t^2 \leq 1} |\overrightarrow{N}| ds dt = \iint\limits_{s^2+t^2 \leq 1} \sqrt{1 + s^2 + t^2} ds dt$$

The last integral can be computed in polar coordinates. In polar coordinates $s = r\cos\theta$, $s = r\sin\theta$ we have $dsdt = rdrd\theta$ and hence

$$\iint\limits_{s^2+t^2\leq 1} \sqrt{1+s^2+t^2}dsdt = \int_0^{2\pi}\int_0^1 \sqrt{1+r^2}rdrd\theta$$

$$= 2\pi \cdot \frac{1}{3}\left(1+r^2\right)^{3/2}\Big|_{r=0}^{r=1}$$

$$= \frac{2}{3}\pi(2^{3/2}-1).$$

The correct answer is (F).

Solution of problem 1.4: By Gauss's theorem we have

$$\iiint\limits_R \sigma dxdydz = \iiint\limits_R \overrightarrow{\nabla}\cdot\overrightarrow{E}dxdydz = \iint\limits_{\partial R} \overrightarrow{E}\cdot d\overrightarrow{S}.$$

The boundary ∂R of the box consists of six rectangular faces each of which is oriented by the outgoing normal vector. The faces can be described as the parts of the box given by the equations $x = 0$, $x = 1$, $y = 0$, $y = 2$, $z = 0$, and $z = 1$. If we write $F_{x=0}$, $F_{x=1}$, etc. for the faces, then we have

$$\iint\limits_{\partial R} \overrightarrow{E}\cdot d\overrightarrow{S} = \iint\limits_{F_{x=0}} \overrightarrow{E}\cdot d\overrightarrow{S} + \iint\limits_{F_{x=1}} \overrightarrow{E}\cdot d\overrightarrow{S}$$

$$+ \iint\limits_{F_{y=0}} \overrightarrow{E}\cdot d\overrightarrow{S} + \iint\limits_{F_{y=2}} \overrightarrow{E}\cdot d\overrightarrow{S}$$

$$+ \iint\limits_{F_{z=0}} \overrightarrow{E}\cdot d\overrightarrow{S} + \iint\limits_{F_{z=1}} \overrightarrow{E}\cdot d\overrightarrow{S}$$

We can analyze each of the six surface integrals separately. If we parametrize the faces $F_{x=0}$ and $F_{x=1}$ by the (y,z)-coordinates, then the normal vector at each point of $F_{x=0}$ is $(-\widehat{\boldsymbol{\imath}})$ and the normal vector at each point of $F_{x=1}$ is $\widehat{\boldsymbol{\imath}}$. Therefore

$$\iint\limits_{F_{x=0}} \overrightarrow{E}\cdot d\overrightarrow{S} = \int_{y=0}^{y=2}\int_{z=0}^{z=1} \overrightarrow{E}(0,y,z)\cdot(-\widehat{\boldsymbol{\imath}})dzdy = \int_{y=0}^{y=2}\int_{z=0}^{z=1} 0dzdy = 0,$$

and

$$\iint\limits_{F_{x=1}} \overrightarrow{E} \cdot d\overrightarrow{S} = \int_{y=0}^{y=2} \int_{z=0}^{z=1} \overrightarrow{E}(1,y,z) \cdot (\widehat{\boldsymbol{\imath}}) dz dy \int_{y=0}^{y=2} \int_{z=0}^{z=1} 0 dz dy = 0.$$

Similarly, if we parametrize the faces $F_{y=0}$ and $F_{y=2}$ by the (x,z) coordinates, then the normal vector at each point of $F_{y=0}$ is $(-\widehat{\boldsymbol{\jmath}})$ and the normal vector at each point of $F_{y=2}$ is $\widehat{\boldsymbol{\jmath}}$. Since $\overrightarrow{E}(x,0,z) \cdot \widehat{\boldsymbol{\jmath}} = 0$ and $\overrightarrow{E}(x,2,z) \cdot \widehat{\boldsymbol{\jmath}} = 0$ it again follows that

$$\iint\limits_{F_{y=0}} \overrightarrow{E} \cdot d\overrightarrow{S} = 0 \text{ and } \iint\limits_{F_{y=2}} \overrightarrow{E} \cdot d\overrightarrow{S} = 0.$$

Finally we have that when we parametrize the faces $F_{z=0}$ and $F_{z=1}$ by the (x,y) coordinates, the normal vector at each point of $F_{z=0}$ is $(-\widehat{\boldsymbol{k}})$ and the normal vector at each point of $F_{y=1}$ is $\widehat{\boldsymbol{k}}$. Since $\overrightarrow{E}(x,y,0) \cdot (-\widehat{\boldsymbol{k}}) = -1$ and $\overrightarrow{E}(x,y,1) \cdot \widehat{\boldsymbol{k}} = 0$ we get that

$$\iint\limits_{F_{z=1}} \overrightarrow{E} \cdot d\overrightarrow{S} = 0,$$

and

$$\iint\limits_{F_{z=0}} \overrightarrow{E} \cdot d\overrightarrow{S} = \int_{x=0}^{x=1} \int_{y=0}^{y=1} (-1) dy dx = -2.$$

Thus the correct answer is (B).

Solution of problem 1.5: The curve C projects to the unit circle in the xy-plane, and from the parametrization

$$\overrightarrow{r}(t) = \langle \cos t, \sin t, 1 + \cos t \rangle, \qquad 0 \leq t < 2\pi$$

we see that this curve can be viewed as the boundary of the part of the plane $z = 1 + x$ that is inside the cylinder $x^2 + y^2 = 1$. Denote this surface by S. We can parametrize S by

$$\overrightarrow{X}(x,y) = \langle x, y, 1 + x \rangle, \qquad \text{for } x^2 + y^2 \leq 1.$$

The normal vector for this parametrization is $\overrightarrow{N}(x,y) = -\widehat{\boldsymbol{\imath}} + \widehat{\boldsymbol{k}}$ and so by Stokes's theorem we have

$$\begin{aligned}
\int_C \overrightarrow{F} \cdot d\overrightarrow{r} &= \iint_S (\overrightarrow{\nabla} \times \overrightarrow{F}) \cdot d\overrightarrow{S} \\
&= \iint_{x^2+y^2\leq 1} \left(\overrightarrow{\nabla} \times \left\langle -y^3, x^3, -z^3 \right\rangle\right) \cdot \langle -1, 0, 1\rangle \, dxdy \\
&= \iint_{x^2+y^2\leq 1} \left\langle 0, 0, 3x^2 + 3y^2 \right\rangle \cdot \langle -1, 0, 1\rangle \, dxdy \\
&= \iint_{x^2+y^2\leq 1} \left(3x^2 + 3y^2\right) dxdy \\
&= \int_0^{2\pi} \int_0^1 3r^2 r dr d\theta \\
&= \frac{3\pi}{2}.
\end{aligned}$$

The correct answer is (D).

Solution of problem 1.6: First find the row reduced echelon form of A. We have

$$\begin{pmatrix} 1 & 2 & 0 & 3 \\ 1 & 2 & -1 & -1 \\ 0 & 0 & 1 & 4 \end{pmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & 1 & 4 \end{pmatrix} \xrightarrow{R_3 \to R_3 + R_2}$$
$$\begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 \to -R_2} \begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

From the row echelon form we see that if we label the variables as x_1, x_2, x_3, x_4, then the free variables are x_2 and x_4 and that $x_1 = -2x_2 -$

$3x_4$, $x_3 = -4x_4$. Thus the system has exactly two free variables. The correct answer is (D).

Solution of problem 1.7: For (i) let

$$A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}.$$

Then

$$A^T A = [a_1, a_2, a_3, a_4, a_5] \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2.$$

So if $A^T A = 0$ it follows that the sum of squares of the entries of A is equal to zero. Since all a_i are real numbers it follows that $a_i^2 \geq 0$ for all $i = 1, \ldots, 5$, and so we must have that $a_i^2 = 0$ for all $i = 1, \ldots, 5$. Therefore (i) is True.

For (ii) note that if A is any 2×3 matrix, then $A^T A$ is a 3×3 matrix, while AA^T is a 2×2 matrix. So (ii) is False.

For (iii) we compute

$$\left(A^{-1}BA\right)^2 = \left(A^{-1}BA\right)\left(A^{-1}BA\right) = A^{-1}BAA^{-1}BA = A^{-1}BIBA = A^{-1}B^2A.$$

So (iii) is True.

Solution of problem 1.8: We can parametrize S by

$$\overrightarrow{X}(s,t) = \left\langle \sqrt{3}\cos t, \sqrt{3}\sin t, s \right\rangle, \qquad 0 \leq s \leq 4,\ 0 \leq t \leq 2\pi.$$

The normal vector for this parametrization is

$$\begin{aligned}\overrightarrow{N}(s,t) &= \frac{\partial \overrightarrow{X}}{\partial s} \times \frac{\partial \overrightarrow{X}}{\partial t} \\ &= \langle 0,0,1\rangle \times \langle -\sqrt{3}\sin t, \sqrt{3}\cos t, 0\rangle \\ &= \langle -\sqrt{3}\cos t, -\sqrt{3}\sin t, 0\rangle.\end{aligned}$$

Note that this vector is pointing inward so the correct orientation o fthe surface will be given by the vector $-\overrightarrow{N}(s,t)$ Now we can compute the surface integral:

$$\begin{aligned}\iint\limits_S \overrightarrow{F} \cdot d\overrightarrow{S} &= \int_0^{2\pi}\int_0^4 \overrightarrow{F} \cdot (-\overrightarrow{N}(s,t)) ds dt \\ &= \int_0^{2\pi}\int_0^4 \langle 2\sqrt{3}\cos t, 2\sqrt{3}\sin t, s\log(1+3\cos^2 t)\rangle \\ &\qquad \cdot \langle \sqrt{3}\cos t, \sqrt{3}\sin t, 0\rangle ds dt \\ &= 6\int_0^{2\pi}\int_0^4 ds dt \\ &= 6 \cdot 2\pi \cdot 4 \\ &= 48\pi.\end{aligned}$$

Therefore the correct answer is (A).

Solution of problem 1.9: Since $\overrightarrow{r}(s,t) = se^t\widehat{\boldsymbol{\imath}} + (s+t)\widehat{\boldsymbol{\jmath}} + e^{st}\widehat{\boldsymbol{k}}$ we have

$$\begin{aligned}\frac{\partial \overrightarrow{r}}{\partial s} &= e^t\widehat{\boldsymbol{\imath}} + \widehat{\boldsymbol{\jmath}} + te^{st}\widehat{\boldsymbol{k}} \\ \frac{\partial \overrightarrow{r}}{\partial t} &= se^t\widehat{\boldsymbol{\imath}} + \widehat{\boldsymbol{\jmath}} + se^{st}\widehat{\boldsymbol{k}}.\end{aligned}$$

The point $(0,0,1)$ on S corresponds to $s=0$, $t=0$. We evaluate

$$\frac{\partial \overrightarrow{r}}{\partial s}(0,0) = \widehat{\boldsymbol{\imath}} + \widehat{\boldsymbol{\jmath}}$$

$$\frac{\partial \overrightarrow{r}}{\partial t}(0,0) = \widehat{\boldsymbol{\jmath}},$$

and so

$$\left[\frac{\partial \overrightarrow{r}}{\partial s}(0,0)\right] \times \left[\frac{\partial \overrightarrow{r}}{\partial t}(0,0)\right] = (\widehat{\boldsymbol{\imath}} + \widehat{\boldsymbol{\jmath}}) \times \widehat{\boldsymbol{\jmath}} = \widehat{\boldsymbol{\imath}} \times \widehat{\boldsymbol{\jmath}} = \widehat{\boldsymbol{k}}$$

The length of this vector is 1 so the the unit normal to S at the point $(0,0,1)$ is $\overrightarrow{n} = \widehat{\boldsymbol{k}}$. The correct answer is (A).

Solution of problem 1.10: Let x denote the first column of X. Then x must solve the system

$$\begin{pmatrix} 4 & 6 \\ 6 & 9 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Using Gaussian elimination we get

$$\left(\begin{array}{cc|c} 4 & 6 & 1 \\ 6 & 9 & 1 \end{array}\right) \xrightarrow{R_2 \to R_2 - \frac{3}{2}R_1}$$

$$\left(\begin{array}{cc|c} 4 & 6 & 1 \\ 0 & 0 & -1/2 \end{array}\right)$$

Thus the rank of the coefficient matrix is 1 whereas the rank of the augmented matrix is 2. This shows that the system for x has no solution, and consequently the system for X has no solution. The correct answer is (A).
