Math 170, Section 002 Spring 2012
Practice Final Exam with Solutions

Contents

1 Problems

Question 1: Consider all natural numbers that can be written only with the digits 1 and 2 and such that the sum of their digits is equal to 13. How many such numbers are there?

(a) 55 (b) 377 (c) 34

(d) 144 (e) 89 (f) 233

Solution Key: 2.1 Solution: 3.1

Question 2: What is 2012^{2012} modulo 7?

(a) 2 (b) 1 (c) 3

(d) 6 (e) 4 (f) 5

Solution Key: 2.2 Solution: 3.2

Question 3: Which of the following numbers are irrational?

(i) $\frac{5\sqrt{3}}{7}$;

(ii) $2(\pi - 1)$;

(iii) $8^{\sqrt{8/72}}$;

(iv) $\left(\sqrt{2}\right)^{\sqrt{16}}$.

(a) **(i)**, **(ii)**, and **(iii)** only

(b) **(ii)** and **(iii)** only

(c) **(i)** and **(ii)** only

(d) **(iii)** and **(iv)** only

(e) **(i)** and **(iv)** only

(f) **(ii)** and **(iv)** only

SOLUTION KEY: 2.3

SOLUTION: 3.3

Question 4: True or False. Give a reason or a counter-example.

- **(1)** The set of all natural numbers that begin with the digits 3141592, and the set of all natural number have the same cardinality.
- **(2)** There exists an irrational number a with the property that there are no rational numbers between a and $a+1$.
- **(3)** The set of all rational numbers with denominators between 1 and 1000, and the set of all rational numbers between 1 and 1000 have the same cardinality.

(a)

(1)	**(2)**	**(3)**
T	T	T

(b)

(1)	**(2)**	**(3)**
T	F	T

(c)

(1)	**(2)**	**(3)**
F	T	F

(d)

(1)	**(2)**	**(3)**
F	F	F

(e)

(1)	**(2)**	**(3)**
T	F	F

(f)

(1)	**(2)**	**(3)**
T	T	F

SOLUTION KEY: 2.4 SOLUTION: 3.4

Question 5: Let B be an object in the 6-dimensional space $\mathbb{R}^6$. Let $(x_1, x_2, x_3, x_4, x_5, x_6)$ denote the coordinates of a point in $\mathbb{R}^6$. Fixing the values of two of the coordinates describes a four dimensional subspace of $\mathbb{R}^6$. Suppose that for any two real numbers a and b we know that if we slice B with the 4-dimensional subspace

$$x_1 = a$$
$$x_5 = b$$

then the resulting slice is a has equation

$$x_2^2 + x_3 + 2x_4^3 - 5x_6^3 = (a + 7b)^2.$$

What is the dimension of B?

(a) 0 (b) 1 (c) 2

(d) 3 (e) 4 (f) 5

SOLUTION KEY: 2.5 SOLUTION: 3.5

Question 6: Consider the letters

A D O M

Which one of the following statements is correct? Explain your reasoning.

(a) A and D are equivalent by distortion

(b) D and O are equivalent by distortion

(c) O and M are equivalent by distortion

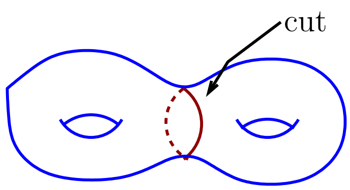
(d) A and O are equivalent by distortion

(e) A and M are equivalent by distortion

(f) D and M are equivalent by distortion

SOLUTION KEY: 2.6 SOLUTION: 3.6

Question 7: Suppose we cut a hollow 2-hole pretzel along the circle going around neck of the pretzel:



Which of the following objects is equivalent by distortion to this cut pretzel?

(a) a sphere with three holes

(b) a torus with two holes

(c) a hemipshere with a hole

(d) two copies of a torus with a hole

(e) a 2-hole pretzel with two holes

(f) a cylinder with two holes

Solution Key: 2.7 Solution: 3.7

Question 8: True or False. Give a reason or a counter-example.

(1) A connected planar graph with 54 vertices, and 177 edges, must have 125 faces.

(2) A planar graph with 7 vertices, 4 edges, and 7 faces must have 8 connected components.

(3) If a planar graph has 2 connected pieces, 12 edges, and 7 vertices, then it splits the plane into 8 regions.

(a)

(1)	**(2)**	**(3)**
T	T	T

(b)

(1)	**(2)**	**(3)**
T	F	T

(c)

(1)	**(2)**	**(3)**
F	T	F

(d)

(1)	**(2)**	**(3)**
F	F	F

(e)

(1)	**(2)**	**(3)**
F	F	T

(f)

(1)	**(2)**	**(3)**
F	T	T

SOLUTION KEY: 2.8 SOLUTION: 3.8

Question 9: The fish population in a small pond is modeled by the Verhulst formula $P_{n+1} = P_n + cP_n(1 - P_n)$, where P_n stands for the fish population density after n years. The pond can sustain 300 fish and has initial fish population of 1200 fish. If after two years the fish population is 600 fish, how big will the fish population be after three years?

(a) 200 fish (b) 300 fish (c) 400 fish

(d) 500 fish (e) 600 fish (f) 700 fish

SOLUTION KEY: 2.9 SOLUTION: 3.9

Question 10: You roll three dice. What is the probability that you will get three numbers that add up to 6?

(a) $\frac{1}{216}$ (b) $\frac{4}{27}$ (c) $\frac{11}{216}$

(d) $\frac{5}{36}$ (e) $\frac{5}{108}$ (f) $\frac{7}{36}$

SOLUTION KEY: 2.10 SOLUTION: 3.10

Question 11: Suppose 4 students out of 100 cheated during the last year, and you used the two-coin method to survey those 100 students. How many students would you expect to answer "yes"?

(a) 4 (b) 11 (c) 18

(d) 21 (e) 24 (f) 27

SOLUTION KEY: 2.11 SOLUTION: 3.11

Question 12: It is known that $8,000$ students go to school in a college town. A local supermarket surveys 720 customers and determines that 180 of them are college students. Based on this information, estimate the non-student population of the town.

(a) $24,000$ (b) $18,000$ (c) $16,000$

(d) $32,000$ (e) $11,000$ (f) $28,000$

SOLUTION KEY: 2.12 SOLUTION: 3.12

Question 13: John deposited \$ 1,600 in savings account. After two years John had \$ 1,681 in the account. What was the annual percentage interest paid by this account?

(a) 2 % (b) 2.7 % (c) 1.4 %

(d) 1.8 % (e) 2.5 % (f) 3.1 %

SOLUTION KEY: 2.13 SOLUTION: 3.13

Question 14: A class of 110 people is voting for a new class president. The voters are choosing among four candidates by using the ranked voting system. Each voter ranks all four candidates: 1, 2, 3, 4 with 1 being the highest ranking. The candidates are then ordered by a total numerical score - the candidate with a lowest numerical score has the highest ranking in the outcome, the candidate with the next numerical score has the second highest ranking, etc..

Which of the following properties of a fair voting system are **not** satisfied for this voting system? Circle all that apply. Explain your reasoning.

(a) **(system rationality)** (b) **(determinism)**

(c) **(consensus)** (d) **(impartiality)**

(e) **(independence of third alternative)** (f) **(no dictators)**

SOLUTION KEY: 2.14 SOLUTION: 3.14

2 Solution key

(1) (f)

(2) (a)

(3) (e)

(4) (b)

(5) (f)

(6) (b)

(7) (d)

(8) (b)

(9) (d)

(10) (e)

(11) (f)

(12) (a)

(13) (d)

(14) (a)

3 Solutions

Solution of problem 1.1: Let A_N be the number of all natural numbers that can be written with the digits 1 and 2 only and such that the sum of all digits is equal to N. We can compute A_N recursively:

- If $N = 1$, then there is only one such number: the number 1. Thus $A_1 = 1$.
- If $N = 2$, then there are two such numbers: 11 and 2. Thus $A_2 = 2$.
- To compute A_N for an arbitrary N, note that any number made out of 1's and 2's only the first digit of the number must be either 1 or 2. If we start with 1, then we can complete the number in A_{N-1} ways, while if we start with 2 we can complete the number in A_{N-2} ways. Therefore $A_N = A_{N-1} + A_{N-2}$.

Using this formula we compute

$$\begin{aligned} &A_1 = 1, \quad A_2 = 2, \quad A_3 = 3, \quad A_4 = 5, \quad A_5 = 8, \quad A_6 = 13, \\ &A_7 = 21, \quad A_8 = 34, \quad A_9 = 55, \quad A_{10} = 89, \\ &A_{11} = 144, \quad A_{12} = 233, \quad A_{13} = 377. \end{aligned}$$

The correct answer is (f). □

Solution of problem 1.2: Dividing 2012 by 7 we get a quotient 287 and remainder 3. Thus 2012 equals 3 modulo 7, and so

$$2012^{2012} = 3^{2012} \quad \text{modulo } 7.$$

Computing the first few powers of 3 modulo 7 we see that

$$\begin{aligned} 3^2 &= 9 = 2 \quad \text{modulo } 7, \\ 3^3 &= 27 = -1 \quad \text{modulo } 7. \end{aligned}$$

In particular since $2012 = 3 \cdot 670 + 2$ we get

$$3^{2012} = (3^3)^{670} \cdot 3^2 = (-1)^{670} \cdot 2 = 2 \quad \text{modulo } 7.$$

The correct answer is (a). □

Solution of problem 1.3: **(i)** Let $q = (5\sqrt{3})/7$. Then the irrational number $\sqrt{3}$ is equal to $(7q)/5$. If q is rational, since a product or ratio of rational numbers is rational, we will get that $\sqrt{3}$ is rational which is a contradicton. Thus $(5\sqrt{3})/7$ is irrational.

(ii) The number $q = 2(\pi - 1)$ can not be rational. Indeed the irrational number π is equal to $1 + q/2$. But a ratio and sum of rational numbers is rational so if q is rational we will get that π is rational, which is a contradiction. Thus $2(\pi - 1)$ is irrational.

(iii) The expression simplifies:

$$8^{\sqrt{8/72}} = 8^{\sqrt{1/9}} = 8^{1/3} = \sqrt[3]{8} = 2.$$

Thus $8^{\sqrt{8/72}}$ is rational.

(iv) The expression simplifies:

$$\left(\sqrt{2}\right)^{\sqrt{16}} = \left(\sqrt{2}\right)^{4} = 2^2 = 4.$$

Thus $\left(\sqrt{2}\right)^{\sqrt{16}}$ is rational.

The correct answer is (c). □

Solution of problem 1.4: **(1)** is True. There is a one-to-one correspondence between the set of all natural numbers and the set of all natural numbers beginning with 3141592. The correspondence is given by matching a number N eginning with 3141592 with the natural number N' obtained from N by deleting the leading digits 3141592.

(2) is False. We argued in class that for every two real numbers $a < b$ there exists a *rational* number c such that $a < c < b$.

(3) False. The set of all rational numbers with denominators between 1 and 1000 is infinite: it contains the natural numbers since they can be viewed as rational numbers with denominators 1. The set of all natural numbers between 1 and 1000 is finite. Thus the two sets can not have the same cardinality.

The correct answer is (b). □

Solution of problem 1.5: The intersection of B with any one of the four dimensional subspaces is given by one constraint so it depends on three parameters. Since we have a two parameter family of four dimensional subspaces, the points in B depend on a tota; pf $3 + 2 = 5$ parameters. This shows that B is 5 dimensional.

This argument can be made also in terms of formulas. We can solve the constraint for x_3 in terms of the other variables. Then a general point in the slice of B by the subspace satisfying $x_1 = a$, $x_5 = b$ is given by

$$(a, x_2, -x_2^2 - 2x_4^3 + 5x_6^3 + (a + 7b)^2, x_4, b, x_6).$$

In other words, the general point of B is described by the five parameters a, b, x_2, x_4, x_6, and so B is five dimensional.

Therefore the correct answer is (f). □

Solution of problem 1.6: Cleraly D can be distorted into O by rounding-off the corners on the left side of D. Thus, the correct answer must be (b). We can also easily rule out the other answers directly. Indeed, if we delete a point from the bottom part of one of the legs of A, we will get two connected pieces. But removing any point from either D or O we get a single piece. Therefore A can not be distorted into either D or O. Similarly, removing a point from M results in two pieces, and so M can not be distorted into either D or O. Finally, if we delete the top apex of A we will get one piece, whereas when we remove any point from M we will get two pieces. So A can not be distorted into M.

The correct answer is (b). □

Solution of problem 1.7: If we cut the two hole pretzel along the indicated circle we will obtain two connected pieces each with one hole and one boundary. Thus the correct choice is (d). □

Solution of problem 1.8: **(1)** A connected planar graph has Euler characteristic 2. So if the graph has 54 vertices, 177 edges, and F faces, then it follows that $54 - 177 + F = 2$, or that $F = 125$. Therefore **(1)** is True.

(2) A planar graph with 7 vertices, 4 edges, and 7 faces has Euler characteristic $7 - 4 + 7 = 10$. By the Euler characteristic theorem such a graph must have $10 - 1 = 9$ connected components. Therefore **(2)** is False.

(3) If a planar graph has 2 connected pieces then it has Euler characteristic 3. If the graph also has 12 edges, 7 vertices, and F faces, then $3 = 7 - 12 + F$. In other words, we must have that $F = 8$. Therefore **(3)** is True.

The correct answer is (b). □

Solution of problem 1.9: Since the pond can sustain 300 fish and the initial population is 1200 fish, it follows that the initial population density is

$$P_1 = \frac{1200}{300} = 4.$$

Similarly, the population density after two years is

$$P_2 = \frac{600}{300} = 2.$$

The Verhulst model gives the relation

$$\frac{P_2}{P_1} = 1 + c(1 - P_1),$$

or equivalently

$$\frac{2}{4} = 1 - 3c.$$

This gives

$$c = \frac{1}{6}$$

Applying againthe Verhulst model we get

$$\frac{P_3}{P_2} = 1 + c(1 - P_2),$$

or equivalently

$$\frac{P_3}{2} = 1 + \frac{1}{6}(1 - 2).$$

Solving for P_3 we get $P_3 = 5/3$. If x is the fish population in three years, then x obeys $x/300 = 5/3$, and so $x = 500$.

The correct answer is (d). □

Solution of problem 1.10: If we roll three numbers that add up to 6, none of those numbers can be bigger than 4. Furthermore if one of the numbers is 4 then the other two must be 1 and 1. Similarly, if one of the numbers is 3, then the other two must be 1 and 2. Finally, if none of the numbers is equal to 3 or 4, then all three numbers must be equal to 2. So the outcomes of the roll that show three numbers adding up to 6 are 411, 141, 114, 312, 321, 132, 231, 123, 213, 222. Since the total number of outcomes of the roll is $6 \cdot 6 \cdot 6 = 216$ the probability of getting three numbers that add to 6 is $10/216 = 5/108$.

The correct answer is (e). □

Solution of problem 1.11: Let x denote the number of students who tossed hh and cheated, and let y denote the number of students who tossed hh and did not cheat. Then we have that roughly $4x = 4$ or $x = 1$. Since the number of people who tossed hh is about $100/4 = 25$ it follows that $y = 25 - 1 = 24$. In particular about $3x + y = 27$ people have said "yes", and about $x + 3y = 73$ people said "no".

The correct answer is (f). □

Solution of problem 1.12: Out of the 720 people surveyed, 180 are college students. Based on this statistical sample the proportion of college students in the town population is $180/720 = 1/4$. Thus about a quarter of the twon population is $8,000$ people. Then the remaining three quarters of the town population are $3 \cdot 8,000 = 24,000$.

The correct answer is (a). □

Solution of problem 1.13: Let I be the annual percentage rate on the account. Then we have

$$1681 = 1600 \left(1 + \frac{I}{100}\right)^2,$$

or

$$I = 100 \left(\sqrt{\frac{1681}{1600}} - 1\right) = \frac{100}{40} = 2.5.$$

The correct answer is (e). □

Solution of problem 1.14: The **(system rationality)** property does not hold for this voting system: ties are possible for the total numerical scores.

The **(determinism)** property holds: only the ranking preferences of the voters are taken into account in determining the outcome of the vote.

The **(consensus)** property holds: if all voters rank A higher than B, then A will have a lower numerical score than B, and so will rank higher than B in the outcome of the vote.

The **(impartiality)** property holds: the voting system remains unchanged if we shuffle the candidates.

The **(independence of a third alternative)** property holds: the relative ranking of candidates A and B is independent of the voters preferences for another candidate X since with or without X the numerical score of A will be lower than the numerical score of B.

The **(no disctators)** property holds: there are many more voters than candidates so a low numerical score by a single voter can not pull a total numerical score down.

The correct answer is (a). □