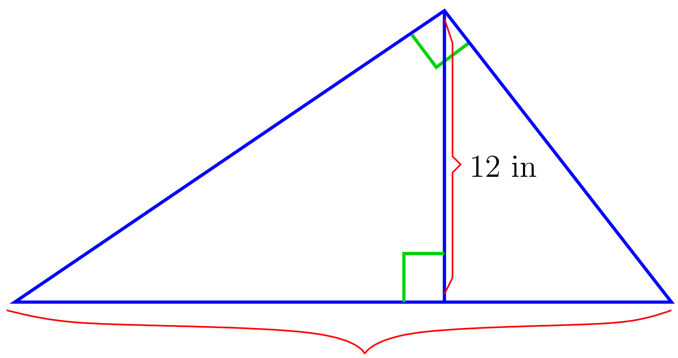
Math 170, Section 002 Spring 2012
Practice Exam 2 with Solutions

Contents

1 Problems

Question 1: A right triangle has hypothenuse of length 25 in and an altitude to the hypothenuse of length 12 in. What is the sum of the lengths of the two sides of the triangle adjacent to the right angle?

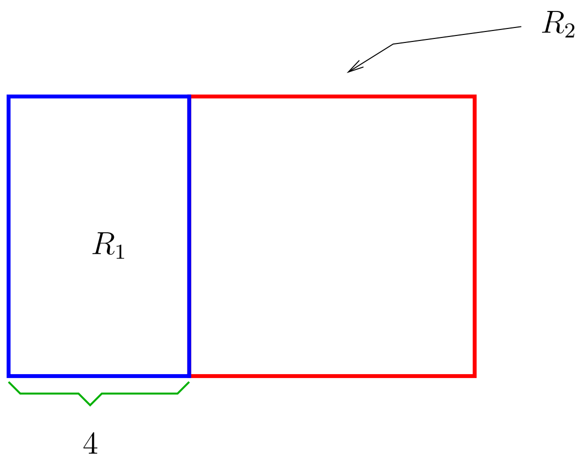


(a) 27 in (b) 24 in (c) 30 in

(d) 35 in (e) $(15 + \sqrt{22})$ in (f) 29 in

Solution Key: 2.1 Solution: 3.1

Question 2: Consider a golden rectangle R_1 whose shorter side has length 4. Let R_2 be the golden rectangle obtained by adjoining a square to the shorter side of R_1.



What is the ratio $\dfrac{\text{area}(R_2)}{\text{area}(R_1)}$?

(a) $1+\varphi$ (b) 4φ (c) φ^3

(d) $\sqrt{5}$ (e) $1/\varphi$ (f) $4/\varphi$

Solution Key: 2.2 Solution: 3.2

Question 3: Slice off the four top vertices of a cube to obtain a solid with different types of faces. How many edges does this new solid have?

(a) 12 (b) 20 (c) 18

(d) 32 (e) 24 (f) 52

Solution Key: 2.3 Solution: 3.3

Question 4: Let B be some object in 15 dimensional space. Suppose we start slicing B with 14-dimensional spaces parallel to a fixed coordinate 14-dimensional subspace. If we know that these cross-sectional slices of B all look like hollow 3-dimensional octahedra, what is the dimension of B?

(a) 3 (b) 8 (c) 14

(d) 15 (e) 4 (f) 5

Solution Key: 2.4 Solution: 3.4

Question 5: Consider the letters

D T N Y

Which one of the following statements is correct? Explain your reasoning.

(a) D and Y are equivalent by distortion

(b) D and N are equivalent by distortion

(c) Y and N are equivalent by distortion

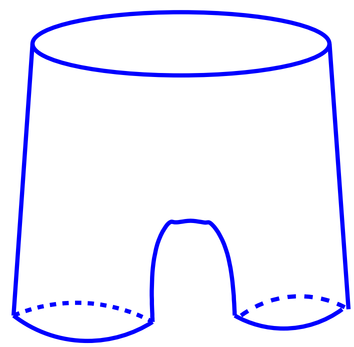
(d) D and T are equivalent by distortion

(e) T and Y are equivalent by distortion

(f) N and T are equivalent by distortion

SOLUTION KEY: 2.5 SOLUTION: 3.5

Question 6: Which of the following objects is equivalent by distortion to the pair of pants:

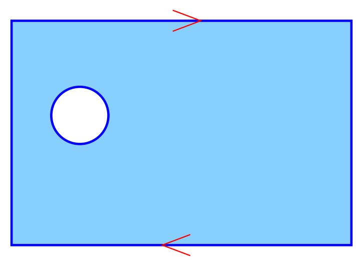


(a) a torus
(b) a sphere
(c) a hemisphere
(d) a torus with one hole
(e) a cylinder
(f) a sphere with three holes

SOLUTION KEY: 2.6

SOLUTION: 3.6

Question 7: The edges of a rectangular sheet of fabric with a circular hole in the middle are identified as shown on the following picture:



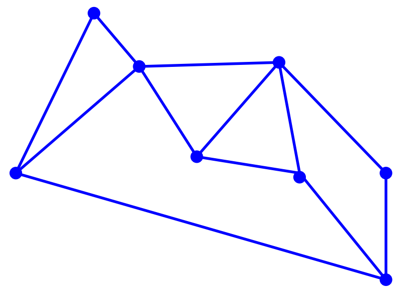
How many boundaries does the resulting surface have?

(a) 0 (b) 1 (c) 2

(d) 3 (e) 4 (f) 5

SOLUTION KEY: 2.7 SOLUTION: 3.7

Question 8: Consider the following graph:



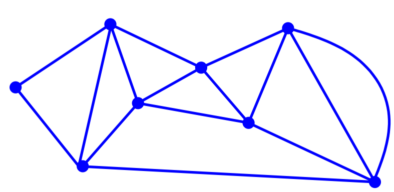
What is the minimal number of edges we must add to this graph so that it contains an Euler circuit? Draw these edges clearly on the picture.

(a) 0 (b) 1 (c) 2

(d) 3 (e) 4 (f) 5

SOLUTION KEY: 2.8 SOLUTION: 3.8

Question 9: Label the consequtive edges of an Euler circuit in the following graph:



SOLUTION KEY: 2.9 SOLUTION: 3.9

Question 10: True or False. Give a reason or a counter-example.

(1) There exists a planar graph with 6 vertices, 10 edges, and 6 faces.

(2) A planar graph consisting of 4 connected pieces must have Euler characteristic 5.

(3) If a connected planar graph has 6 edges and splits the plane into 7 regions, then it must have 1 vertex.

(a)

(1)	(2)	(3)
T	T	T

(b)

(1)	(2)	(3)
T	F	T

(c)

(1)	(2)	(3)
F	T	F

(d)

(1)	(2)	(3)
F	F	F

(e)

(1)	(2)	(3)
T	F	F

(f)

(1)	(2)	(3)
T	T	F

Solution Key: 2.10 Solution: 3.10

2 Solution key

(1) (d)

(2) (a)

(3) (e)

(4) (a)

(5) (e)

(6) (f)

(7) (c)

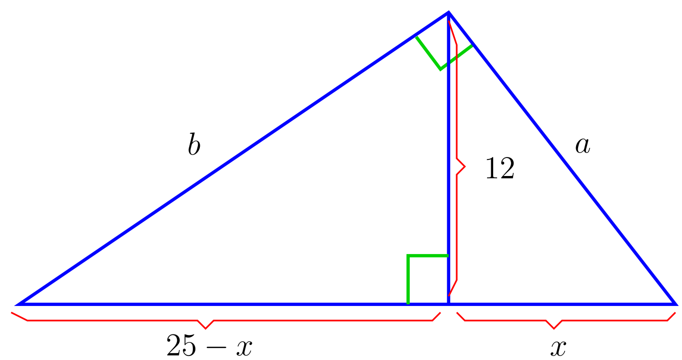
(8) (c)

(9) see solution

(10) (a)

3 Solutions

Solution of problem 1.1: The altitude splits teh hypothenuse into two segments. If we denote the length of the shorter segment by x, then the length of the other segment will be $25 - x$.



If we denote the corresponding sides of the triangle by a and b respectively we can apply the Pythagorean theorem to two small triangles and to the big triangle:

$$\begin{aligned} x^2 + 12^2 &= a^2 \\ (25 - x)^2 + 12^2 &= b^2 \\ a^2 + b^2 &= 25^2 \end{aligned}$$

Substituting the first two identities into the third one we get

$$x^2 + (25 - x)^2 + 288 = 625$$

or

$$x^2 + 625 - 50x + x^2 + 288 = 625.$$

After cancelling 625 from both sides and dividing both sides by 2 we get the equation

$$x^2 - 25x + 144 = 0.$$

By the quadratic formula we have

$$x = \frac{25 \pm \sqrt{25^2 - 4 \cdot 144}}{2} = \frac{25 \pm \sqrt{49}}{2} = \frac{25 \pm 7}{2}.$$

Thus x must be equal to either 9 or 16 , and since x was the shorter of the two segments we must have $x = 9$. Substituting this in the two identities involving a and b above we get

$$\begin{aligned} a^2 &= x^2 + 12^2 = 81 + 144 = 225, \\ b^2 &= (25 - x)^2 + 12^2 = 256 + 144 = 400. \end{aligned}$$

Therefore we have $a = 15$, $b = 20$, and $a + b = 35$. The correct answer is (d). □

Solution of problem 1.2: Denote the longer side of R_1 by x. Since R_1 is a golden rectangle we have that $x/4$ must be equal to the golden ratio φ, i.e.

$$x = 4\varphi.$$

In particular R_1 has sides 4 and 4φ, and R_2 has sides 4φ and $4 + 4\varphi$. We can now compute the ratio of areas:

$$\frac{\text{area}(R_2)}{\text{area}(R_1)} = \frac{4\varphi \cdot (4 + 4\varphi)}{4 \cdot 4\varphi} = \frac{16\varphi(1 + \varphi)}{16\varphi} = 1 + \varphi.$$

The correct answer is (a). □

Solution of problem 1.3: Slicing off a vertex of cube creates a new triangular face. Also, when we slice off a vertex all the original edges remain edges and we introduce three new edges - the sides of teh newly created triangular face. Therefore, when we slice off the four top edges we introduce $4 \cdot 3 = 12$ new edges. Adding these to the original 12 edges the cube has we get a total of 24 edges for the new solid. The correct answer is (e). □

Solution of problem 1.4: We have a stack of 14 dimensional spaces making-up the 15 dimensional space. In each of these 14-dimensional spaces we have a cross-section of B which is a hollow 3-dimensional octahedron, and so is 2-dimensional. In other words B is a stack of 2-dimensional objects. The only additional parameter in B is the thickness of the stack. So B is $2 + 1 = 3$-dimensional. The correct answer is (a). □

Solution of problem 1.5: D can not be distorted into T. Indeed, if there was such a distortion, then it will distort T with any point deleted into D with some point deleted. But if we delete the crossing point of T, we will be left with an object consisting of three connected pieces, while deleting any point from D leaves a single piece. Three connected pieces can not be distorted continuously into a single connected piece, so D can not be distorted into T. For the same reason D can not be distorted into Y.

Similarly D can not be distorted into N. Indeed - removing any point from D leaves a single connected piece, while removing any point from N leaves two connected pieces.

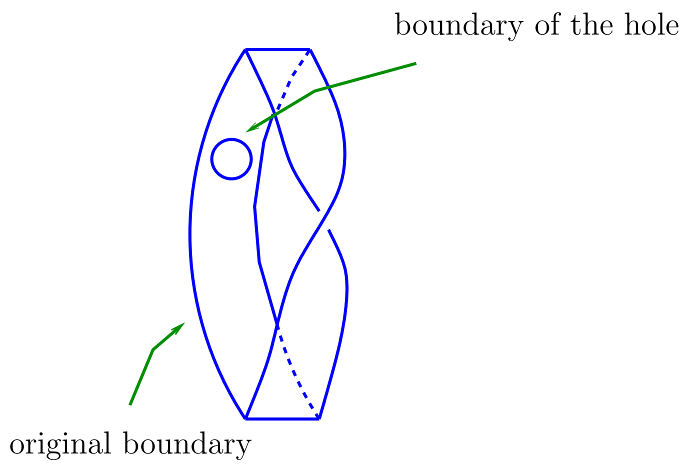
Also neither Y nor T can be distorted into N. Removing the crossing point from either Y or T will leave three connected pieces, while removing any point from N leaves two connected pieces.

Finally Y can clearly be distorted into T by taking the two top portions of Y and pushing them continuously down until we get a horizontal cross bar.

Therefore the correct answer is (e). □

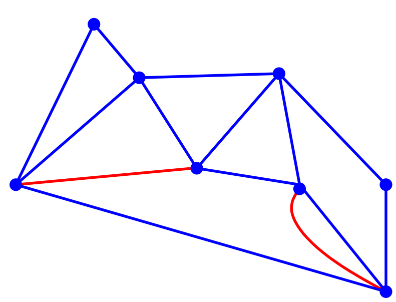
Solution of problem 1.6: The pairs of pants has three boundary circles. A torus and a sphere have no boundary so they can not be topologically equivalent to a pair of pants. Similarly a hemisphere and a torus with one hole have one boundary circles, so they are not equivalent to a pair of pants. A cylinder has two boundary circles so it can not be topologically equivalent to a pair of pants. So the only possibility is the sphere with three holes which indeed has three boundary circles. The correct answer is (f). □

Solution of problem 1.7: A rectangular sheet of fabric with two parallel boundaries identified with a flip of orientation produces a Möbius band, and so has a single boundary. Our sheet of fabric has an additional circular hole and so after the identification we will get a Möbius band with a hole:



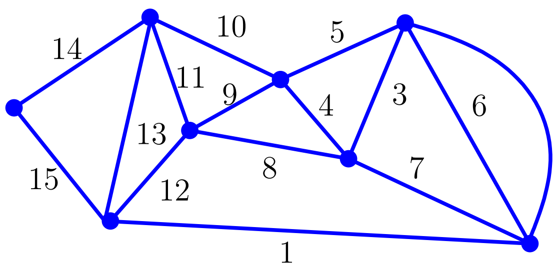
This surface has two boundary edges: the original single boundary edge of the Möbius band and the new boundary edge of the circular hole. Thus the correct choice is (c). □

Solution of problem 1.8: The graph has four 3-valent vertices and four vertices of even valency. By Euler circuit theorem a connected graph admits an Euler circuit if and only if all vertices have even valency. So we must add some edges to convert the 3-valent vertices into vertices of even valency. Since we have four 3-valent vertices we can connect them in pairs by two new edges. Since one edge can not connect four vertices, two is the minimal number of edges we can add. One possible way to do this is

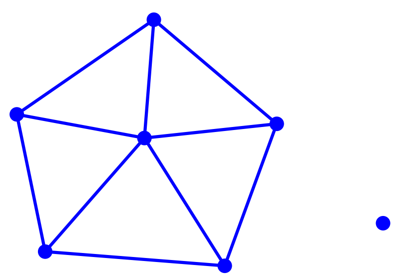


The correct answer is (c). □

Solution of problem 1.9:



Solution of problem 1.10: **(1)** is **True**. For instance the following *disconnected* graph has the required properties:



(2) is **True**. Each of the connected pieces of G has Euler characteristic 2 by the Euler characteritic theorem. Denote the connected pieces of our graph by G_1, G_2, G_3, and G_4. We can build G in stages by starting with G_1 and adding to it the other pieces one at a time. It is convenient to introduce notation for the resulting graph at each stage:

- G_1 the graph at the first stage;
- G_{12} the graph at the second stage, i.e. the graph consisting of the connected pieces G_1 and G_2;
- G_{123} the graph at the third stage, i.e. the graph consisting of the connected pieces G_1, G_2, and G_3;

G the final graph, i.e. the graph consisting of the connected pieces G_1, G_2, G_3, and G_4.

The Euler characteristic of G_1 is 2. If we add to it the piece G_2 then since G_1 and G_2 are disjoint the resulting graph G_{12} has $V(G_1)+V(G_2)$ vertices and $E(G_1)+E(G_2)$ edges. For the regions in which G_1 and G_2 split the plane we have two possibilities: either G_1 is entirely contained in one of the regions for G_2, or G_2 is entirely contained in one of the regions for G_1. In the first case the outside region of G_1 is the same as the region of G_2 in which G_1 is contained, in the second case the outside region of G_2 is the same as the region of G_1 in which G_2 is contained. Hence G_{12} splits theplane in $F(G_1) + F(G_2) - 1$ regions. Therefore the Euler characteristic of of G_{12} is one less than the sum of the Euler characteristics of G_1 and G_2. In other words G_{12} has Euler characteristic $2 + 2 - 1 = 3$.

Next observe that by the same reasoning when we add G_3 to G_{12}, the resulting graph G_{123} will have $V(G_{12})+V(G_3)$ vertices, $E(G_{12})+E(G_3)$ edges, and splits the plane into $F(G_{12}) + F(G_3) - 1$ regions. Therefore the Euler characteristic of G_{123} is one less than the sum of the Euler characteristics of G_{12} and G_3. Thus G_{123} has Euler characteristic $3 + 2 - 1 = 4$.

Finally applying the same argument one more time we conclude that the Euler characteristic of G is one less than the sum of the Euler characteristics of G_{123} and G_4. Thus G has Euler characteristic $4+2-1 = 5$.

(3) is **True**. Suppose that a connected planar graph with 6 edges splits the plane into 7 regions. The Euler characteristic of this graph is $V - 6 + 7 = V + 1$ by definition. BUt by the Euler characteristic theorem the euler characteristic of the graph must be equal to 2. So $V = 1$.

The correct answer is (c). $\square$
