Math 170, Section 002 Spring 2012
Practice Exam 1 with Solutions

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1 Problems

Question 1: After a calculus test the instructor announces that no student got more than four problems wrong. If thirty five people took the test, and the test was graded with no partial credit, at least how many people in the class got the same score on the test?

(a) 7 (b) 4 (c) 6

(d) 9 (e) 21 (f) 18

SOLUTION KEY: 2.1 SOLUTION: 3.1

Question 2: John is climbing a stairway of eleven stairs. If with each step he is climbing either one or two stairs, in how many ways can he climb the stairway.

(a) 89 (b) 72 (c) 55

(d) 77 (e) 91 (f) 144

SOLUTION KEY: 2.2 SOLUTION: 3.2

Question 3: Decompose the number 175 as a sum of non-consequtive Fibonacci numbers. What is the sum of the smallest and the biggest of these Fibonacci numbers.

(a) 94 (b) 146 (c) 149

(d) 63 (e) 91 (f) 78

SOLUTION KEY: 2.3 SOLUTION: 3.3

Question 4: Let p be a natural number with the following properties:

- $p < 100$;
- p is prime and the sum of its digits is 16;
- the natural number obtained by writing the digits of p in the opposite order is also prime.

What is the product of the digits of p?

(a) 36 (b) 28 (c) 63

(d) 21 (e) 45 (f) 7

SOLUTION KEY: 2.4 SOLUTION: 3.4

Question 5: What is the highest power of a prime that appears in the prime decomposition of 3240?

(a) 1 (b) 2 (c) 3

(d) 4 (e) 5 (f) 6

Solution Key: 2.5 Solution: 3.5

Question 6: What is the value of the sum $1 + 3 + 3^2 + \cdots + 3^{15}$ modulo 7?

(a) 0 (b) 1 (c) 3

(d) 4 (e) 5 (f) 6

Solution Key: 2.6 Solution: 3.6

Question 7: The UPC code

$$0\ 78001\ 26\square 05\ 2$$

was scanned incompletely. Find the missing digit.

(a) 0 (b) 3 (c) 1

(d) 8 (e) 7 (f) 9

SOLUTION KEY: 2.7 SOLUTION: 3.7

Question 8: Which of the following numbers are irrational?

(i) $2\sqrt{5} - 1$;

(ii) $\sqrt{\dfrac{54}{150}}$;

(iii) $3\sqrt{\pi}$;

(iv) $-\dfrac{2}{17}$.

(a) **(i)** only (b) **(ii)** only (c) **(ii)** and **(iv)** only

(d) **(iii)** only (e) **(i)** and **(iii)** only (f) **(iii)** and **(iv)** only

SOLUTION KEY: 2.8 SOLUTION: 3.8

Question 9: True or False. Give a reason or a counter-example.

(1) The set of all natural numbers ending in 7 and the set of all negative integers have the same cardinality.

(2) Let A, B, and C be sets of rational numbers, and suppose we know that A has the same cardinality as B, and A has the same cardinality as C. Then B and C can have different cardinalities.

(3) There are infinitely many rational numbers of whose denominators are powers of 3 and which are between 0 and 1.

(a)

(1)	**(2)**	**(3)**
T	T	T

(b)

(1)	**(2)**	**(3)**
T	F	T

(c)

(1)	**(2)**	**(3)**
F	T	F

(d)

(1)	**(2)**	**(3)**
F	F	F

(e)

(1)	**(2)**	**(3)**
T	F	F

(f)

(1)	**(2)**	**(3)**
T	T	F

SOLUTION KEY: 2.9 SOLUTION: 3.9

Question 10: Consider the following sets of real numbers :

- The set A of all natural numbers;
- The set B of all natural numbers numbers that can be written only with the digit 5;
- The set C of all real numbers between 0 and 1 having only 3's and 5's after the decimal point.

Which of the following statements is correct?

(a) A and B have different cardinalities

(b) B and C have the same cardinality

(c) A has a smaller cardinality than C

(d) C has a smaller cardinality than B

(e) A and C have the same cardinality

(f) B has a smaller cardinality than C and C has a smaller cardinality than A

Solution Key: 2.10 Solution: 3.10

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2 Solution key

(1) (a)

(2) (f)

(3) (b)

(4) (c)

(5) (d)

(6) (e)

(7) (b)

(8) (e)

(9) (b)

(10) (c)

3 Solutions

Solution of problem 1.1: Since the largest number of wrong answers was 4 it follows that each student in the class got 0, 1, 2, 3, or 4 problems wrong. Since we have 5 possibilities for the number of mistakes and 35 students, the pigeonhole principle implies that at least 7 students made the same number of mistakes. Indeed if we have 5 rooms labeled by the numbers 0, 1, 2, 3, 4, and we ask each student to go to the room labeled with the number of mistakes he or she made on the test, then at least one room will end up with 7 people at the end. The correct answer is (a).

Solution of problem 1.2: Let us ask a more general question: in how many ways can John climb a stairway of n stairs? Denote the number of ways in which John could climb a stairway of n stairs by J_n.

When the stairway has a relatively few stairs it is easy to find the number of ways. For instance:

$n = 1$ If John has to climb only one stair, then he can do that in only one way. So $J_1 = 1$.

$n = 2$ If John has to climb a stairway of two stairs, tehn he can do it either one at a time, or the two at once. So he can climb the stairway at $J_2 = 2$ ways.

Suppose now John is climbing a stairway with $n > 2$ stairs. There are precisely two different ways in which he can begin the climb: either climb one stair first, or climb two stairs first. If with his first step he climbs one stair, then he is left with climbing a stairway of $n-1$ stairs, which he can climb in J_{n-1} ways. If with his first step he climbs two stairs, then he is left with climbing a stairway of $n-2$ stairs, which he can climb in J_{n-2} ways. Therefore John can climb a stairway of n stairs in exactly $J_{n-1} + J_{n-2}$ ways.

In other words $J_n = J_{n-1} + J_{n-2}$. This shows that J_n is the $n+1$ Fibonacci number. Hence

$$J_1 = 1, \ J_2 = 2, \ J_3 = 3, \ J_4 = 5, \ J_5 = 8, \ J_6 = 13,$$
$$J_7 = 21, \ J_8 = 34, \ J_9 = 55, \ J_{10} = 89, \ J_{11} = 144.$$

The correct answer is (f).

Solution of problem 1.3: The biggest Fibonacci number which is less than 175 is 144. So $175 = 144 + 31$. The biggest Fibonacci number which is less than 31 is 21. So $175 = 144 + 21 + 10$. The biggest Fibonacci number less than 10 is 8, and so $175 = 144 + 21 + 8 + 2$ which are all Fibonacci numbers. The sum of the smallest and biggest of those is $2 + 144 = 146$. The correct answer is (b). $\square$

Solution of problem 1.4: Clearly p must be a two digit prime number since if it is a one digit prime, then the sum of its digits can not be 16. Since the two digits of p add to 16, each of these digits must be at least 7. So the digits of p are among the numbers 7, 8, 9. But we can not have 8 as a digit of p. Indeed 8 can not be the last digit of p since then p will be even and hence not prime. Similarly 8 can not be the first digit of p since then 8 will be the last digit of q, and so q will be even and not prime. The only option then is for the digits of p to be 7 and 9. By inspection we see that both 79 and 97 are prime. So we have that either $p = 79$ or $p = 97$. In either case the product of the digits of p is 63. The correct answer is (c).

Solution of problem 1.5: First we find the prime decomposition of 3240. Since 3240 is even we can divide it by 2: $3240 = 2 \cdot 1620 = 2 \cdot 2 \cdot 810 = 2 \cdot 2 \cdot 2 \cdot 405$. 405 is divisible by 5 so we have $3240 = 2^3 \cdot 405 = 2^3 \cdot 5 \cdot 81 = 2^3 \cdot 3^4 \cdot 5$. Therefore the correct answer is (d). $\square$

Solution of problem 1.6: By the formula for the geometric progression we know that

$$1 + 3 + 3^2 + \cdots + 3^{15} = \frac{3^{16} - 1}{3 - 1} = \frac{3^{16} - 1}{2}.$$

We have that $3^3 = 27 = 4 \cdot 7 - 1$, i.e.

$$3^3 = -1 \bmod 7.$$

This gives

$$\begin{aligned}3^{16} &= (3^3)^5 \cdot 3\\ &= (-1)^5 \cdot 3 \bmod 7\\ &= -3 \bmod 7.\end{aligned}$$

In particular

$$3^{16} - 1 = -3 - 1 = -4 \bmod 7,$$

and so

$$(3^{16} - 1)/2 = -4/2 = -2 = 5 \bmod 7.$$

The correct answer is (d). $\square$

Solution of problem 1.7: Denote the missing digit by x. Then the check-sum formula for the barcode 0 78001 26□05 2 reads

$$3 \cdot 0 + 7 + 3 \cdot 8 + 0 + 3 \cdot 0 + 1 + 3 \cdot 2 + 6 + 3x + 0 + 3 \cdot 5 + 2 = 0 \bmod 10.$$

Simplifying we get

$$3x + 1 = 0 \bmod 10,$$

or equivalently

$$3x = 9 \bmod 10.$$

Thus $x = 3$ and so correct choice is (b).

Solution of problem 1.8: **(i)** Suppose $Q := 2\sqrt{5} - 1$ is rational. Then $\sqrt{5} = \dfrac{Q+1}{3}$. Since 1 and 3 are rational numbers, and since arithmetic operations on natural numbers produce rational numbers, we conclude that $\sqrt{5}$ is rational. This is a contradiction, since we have shown that $\sqrt{5}$ is irrational. Thus $2\sqrt{5} - 1$ is irrational.

(ii) Before we decide whether the number is irrational, let us reduce the fraction under the square root:

$$\frac{54}{150} = \frac{27 \cdot \cancel{2}}{75 \cdot \cancel{2}} = \frac{27}{75} = \frac{9 \cdot \cancel{3}}{25 \cdot \cancel{3}} = \frac{9}{25}.$$

Now we can simplify:

$$\sqrt{\frac{54}{150}} = \sqrt{\frac{9}{25}} = \frac{\sqrt{9}}{\sqrt{25}} = \frac{3}{5}.$$

Thus $\sqrt{\frac{54}{150}}$ is rational.

(iii) Suppose $Q := 3\pi$ is rational. Then $\pi = Q/3$ will be rational which is a contradiction. So 3π is irrational.

(iv) $-2/17$ is the ratio of two integers, so $-2/17$ is rational.

The correct answer is (e).

Solution of problem 1.9: **(1)** is True. We can construct a one-to-one correspondence

$$\left\{\begin{matrix}\text{all natural numbers} \\ \text{ending with 7}\end{matrix}\right\} \longleftrightarrow \{\text{all negative integers}\}$$

explicitly by pairing:

$$\begin{bmatrix}\text{a natural number } n \\ \text{ending with 7}\end{bmatrix} \longleftrightarrow \begin{bmatrix}\text{the negative integer} \\ -(n-7)/10\end{bmatrix}.$$

(2) is False. If we have a one-to-one correspondence f between A and B and we have a one-to-one correspondence g between A and C, then we can construct a one-to-one correspondence between B and C. Indeed, let x be a number in B, then f pairs b with a unique element y in A. On the other hand g will pair y with a unique element z in C. So we can define a correspondence h between B and C by pairing x and z. The correspondence h will be automatically a one-to-one correspondence because of the uniquencess of the pairings f and g.

(3) is True. A rational numbers whose denominator is a power of 3 can be written uniquely as a reduced fraction

$$\frac{m}{3^n}$$

where n is an integer with $n \geq 0$, and m is an integer that is not divisble by 3. Such a number is between 0 and 1 if and only if $0 \leq m \leq 3^n$. So for instance the numbers whose numerators are all equal to 1 and whose denominators are powers of 3 are in our set. But we can easily construct a one-to-one correspondence

$$\left\{\begin{array}{l}\text{all natural numbers}\\ \text{of the form}\\ \\ \quad \dfrac{1}{3^n}, \quad n = 1, 2, \ldots\end{array}\right\} \longleftrightarrow \{\text{all natural numbers}\}$$

by pairing $1/3^n$ with n. Therefore the set of all numbers of the form $1/3^n$ is infinite. But our set contains the set of all of the form $1/3^n$ is infinite, so it also must be infinite.

Solution of problem 1.10: The sets A and B have the same cardinality: we can match a natural number n with the number that is made out of n digits, each of which is equalto 5. The set C has a bigger cardinality. We can see this by Cantor's diagonalization argument. Indeed, suppose that we can find a one-to-one correspondence between the set C and all natural numbers. Let c_n denote the number in C that corresponds to the natural number n. We will construct a number a in C that is not equal to any one of the c_n's. Define a to be the number between 0 and 1 for which:

$$\left[\begin{array}{l}n\text{-th digit of } a \text{ after}\\ \text{the decimal point}\end{array}\right] = \begin{cases} 3 & \begin{array}{l}\text{if the } n\text{-th digit of } c_n \text{ after}\\ \text{the decimal point is 5;}\end{array} \\ \\ 5 & \begin{array}{l}\text{if the } n\text{-th digit of } c_n \text{ after}\\ \text{the decimal point is 3.}\end{array}\end{cases}$$

This defines a completely and ensures that a and c_n have a different n-th digit after the decimal place. Therefore a can not be equal to c_n for any n. Which is a contradiction since $a \in C$ and the c_n's enumerated all the numbers in C.

This shows that C has a bigger cardinality than A (and hence than B). The correct answer is (c).
