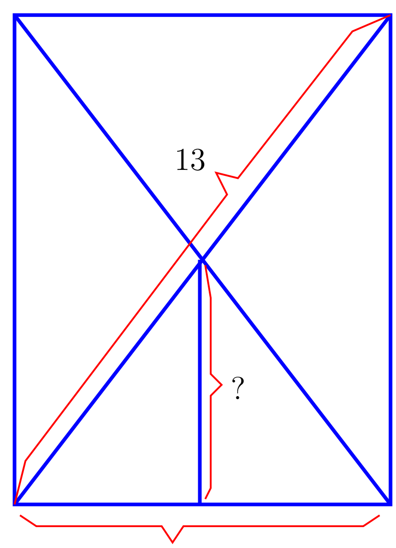
Solutions to the Second Midterm Exam, Math 170, Section 002 Spring 2012

Multiple choice questions.

Question 1. Suppose we have a rectangle with one side of length 5 and a diagonal of length 13. What is the length of the segment joining the midpoint of the side of length 5 and the midpoint of the diagonal?



(a) 4 (b) 7 (c) 5

(d) 12 (e) 6 (f) 10

Answer 1. Let x be the length of the segment joining the midpoint of the side of length 5 and the midpoint of the diagonal. Then x is a side of a right triangle whose other side is of length 5/2 and whose hypothenuse is of length 13/2. By the Pythagorean theorem we have

$$x^2 + \left(\frac{5}{2}\right)^2 = \left(\frac{13}{2}\right)^2,$$

or equivalently

$$x^2 = \left(\frac{13}{2}\right)^2 - \left(\frac{5}{2}\right)^2 = \frac{169-25}{4} = \frac{144}{4} = 36.$$

Thus $x = 6$. The correct choice is (e). □

Question 2. Let G_1 be a golden rectangle whose longer side has length 10. Let G_2 be the golden rectangle obtained from G_1 by removing the largest possible square. Compute the ratio $\text{area}(G_2)/\text{area}(G_1)$.

(a) $1/\varphi$ (b) $10/\varphi$ (c) $10 + \varphi$

(d) $\varphi^2 - 10$ (e) $10(\varphi + 1)$ (f) $1/(1 + \varphi)$

Answer 2. Since G_1 is a golden rectangle it has sides 10 and $10/\varphi$. The largest square that we can inscribe in G_2 is $(10/\varphi) \times (10/\varphi)$ and so G_1 has sides $10/\varphi$ and $10/(\varphi^2)$. Therefore

$$\text{area}(G_1) = 10 \cdot \frac{10}{\varphi} = \frac{100}{\varphi},$$

$$\text{area}(G_2) = \frac{10}{\varphi} \cdot \frac{10}{\varphi^2} = \frac{100}{\varphi^3}.$$

Therefore

$$\frac{\text{area}(G_2)}{\text{area}(G_1)} = \frac{100/(\varphi^3)}{100/\varphi} = \frac{1}{\varphi^2}.$$

In view of the standard identity

$$\varphi^2 = 1 + \varphi$$

this is the same

$$\frac{\text{area}(G_2)}{\text{area}(G_1)} = \frac{1}{1+\varphi}$$

The correct answer is (f). □

Question 3. Slice off three of the top four vertices of a cube to obtain a new solid with various types of faces. Let F be a face of this solid that has the largest number of sides. How many sides does F have?

(a) 5 (b) 6 (c) 7

(d) 8 (e) 10 (f) 12

Answer 3. When we cut off three of the top four vertices of the cube we get a solid with faces of the following types:

- a square (the bottom face of the cube);
- three triangle faces (the three faces left in place of the sliced off vertices);
- two pentagon faces (the two side square faces that had one vertex cut off);
- two hexagon faces (the two side square faces that had two vertices cut off);
- one 7-gon face (the top square face that had three vertice cut off).

So the face with the maximal number of sides is the 7-gon. The correct answer is (c). $\square$

Question 4. An object B in 5 dimensional space consists of all points $(x_1, x_2, x_3, x_4, x_5)$ in $\mathbb{R}^5$ satisfying

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 7 \\ x_1 + x_5 &= 1 \end{aligned}$$

What is the dimension of B?

(a) 0 (b) 1 (c) 2

(d) 3 (e) 4 (f) 5

Answer 4. A point in $\mathbb{R}^5$ is described by the five parameters $(x_1, x_2, x_3, x_4, x_5)$. A point in the object B satisfies

$$\begin{aligned} x_1 &= 1, \\ x_3 &= 7, \\ x_1 + x_5 &= 1, \end{aligned}$$

so for such a point, the parameter x_1 is fixed to have value 1, and the parameter x_3 is fixed to have value 7. Also, if we take into account that $x_1 = 1$ and substitute that in the third equation, we get $1 + x_5 = 1$, i.e. $x_5 = 0$. In other words, B consists of all points in $\mathbb{R}^5$ of the form $(1, x_2, 7, x_4, 0)$. This means that the points of B are all described by two parameters: x_2 and x_4. Thus B has dimension 2 and so the correct answer must be (c). □

Question 5. Consider the letters

N X Y Z

Which one of the following statements is correct? Explain your reasoning.

(a) N and X are equivalent by distortion

(b) N and Y are equivalent by distortion

(c) N and Z are equivalent by distortion

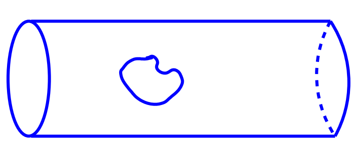
(d) X and Y are equivalent by distortion

(e) X and Z are equivalent by distortion

(f) Y and Z are equivalent by distortion

Answer 5. If we delete the juncture point from X we get a figure with 4 connected pieces. If we delete any point from either N or Z we will get two connected pieces. If we delete a point from Y we can get either 2 or 3 connected pieces. Therefore X can not be topologically equivalent to any of N, Z, or Y. By the same reasoning Y can not be topologically equaivalent to N or Z. Finally N and Z are clearly equivalent by distortion sunce we can distor each of them into a straight segment. The correct answer is (c). □

Question 6. Which of the following objects is topologically equivalent to a cylinder with one hole:



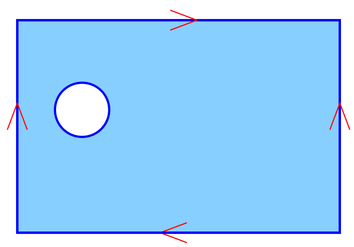
(a) a sphere with three holes (b) a torus with a hole

(c) a hemisphere (d) a hemisphere with a hole

(e) a cylinder (f) a sphere

Answer 6. A cylinder has two boundaries - the two rims of the cylinder. If we cut an extra hole in the surface of the cylinder we will get another boundary. Out of the six listed surfaces only the sphere with three holes has three boundaries. The correct choice is (a). □

Question 7. The edges of a rectangular sheet of fabric with a circular hole in the middle are identified as shown on the following picture:



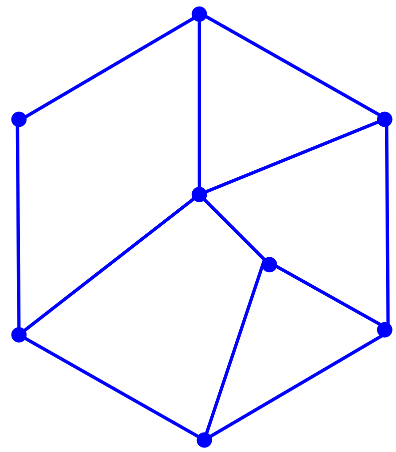
How many boundaries does the resulting surface have?

(a) 0 (b) 1 (c) 2

(d) 3 (e) 4 (f) 5

Answer 7. If we identify the edges of a plain rectangular sheet in the indicated manner we will get a Klein bottle which has no boundaries. The extra hole we have cut out creates an additional boundary so we have a surface with one boundary. The correct answer is (b). □

Question 8. Consider the following graph:



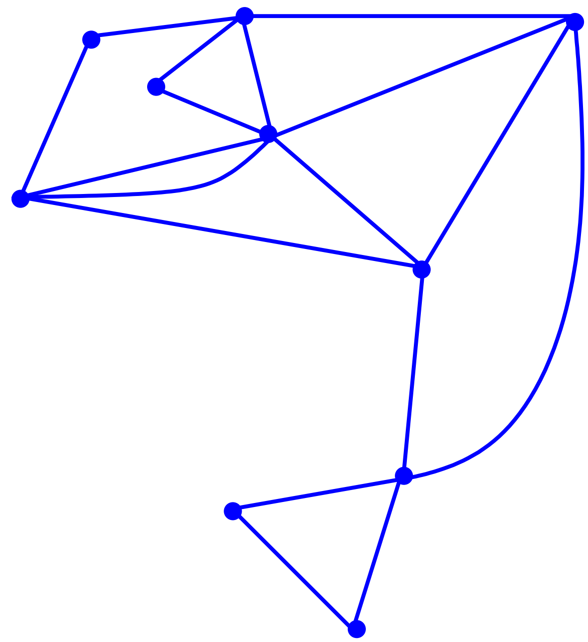
What is the minimal number of edges we must add to this graph so that it contains an Euler circuit? Draw these edges clearly on the picture.

(a) 0 (b) 1 (c) 2

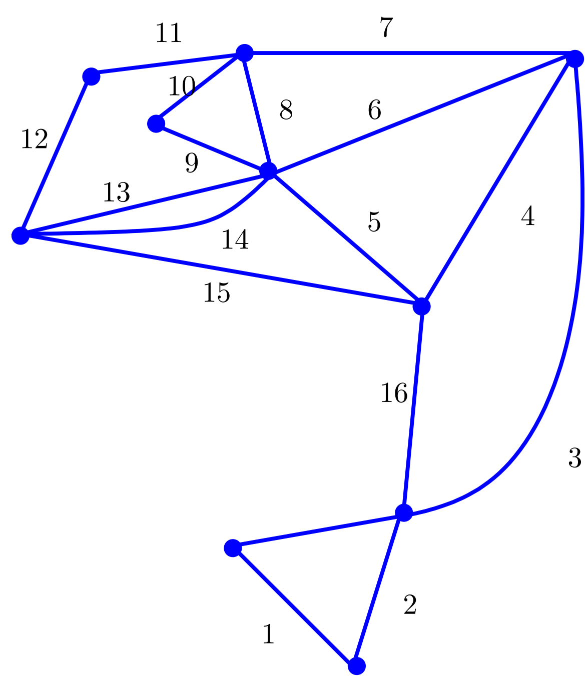
(d) 3 (e) 4 (f) 5

Answer 8. The graph has 6 vertices of valence 3, one vertex of valence 2 and one vertex of valence 4. To have an Euler circuit it is necessary and sufficient to have all vertices be of even valence. So we do not need to do anything to the vertices of valence 2 and 4. So, to guarantee that we have an Euler circuit, we must add edges connecting the odd valenced vertices so that at the end all vertices have even valence. The most economic way to do this is to join the 6 odd valenced vertices in pairs. We need three new edges for this. The correct answer is (d). □

Question 9. Label the consequtive edges of an Euler circuit in the following graph:



Answer 9.



□

Question 10. True or False. Give a reason or a counter-example.

(1) There exists a connected planar graph with 6 vertices, 12 edges, and 6 faces.

(2) A planar graph consisting of 3 connected pieces must have Euler characteristic 3.

(3) If a connected planar graph has 8 edges and 3 vertices, then it splits the plane into 7 regions.

(a)

(1)	**(2)**	**(3)**
T	T	T

(b)

(1)	**(2)**	**(3)**
T	F	T

(c)

(1)	**(2)**	**(3)**
F	T	F

(d)

(1)	**(2)**	**(3)**
F	F	F

(e)

(1)	**(2)**	**(3)**
F	F	T

(f)

(1)	**(2)**	**(3)**
T	T	F

Answer 10. **(1)** The Euler characteristic of a connected planar graph is equal to two. The Euler characteristic of a graph with 6 vertices, 12 edges, and 6 faces is 0. So there can not be such a connected planar graph. Thus **(1)** is False.

(2) By the Euler characteristic theorem, a planar graph with 3 corrected pieces is $3 + 1 = 4$. So **(2)** is also False.

(3) If a connected planar graph has 8 edges, 3 vertices, and F faces, then by the Euler characteristic theorem we have $3 - 8 + F = 2$ or $F = 7$. So **(3)** must be True.

The correct answer is (e). $\square$