

Solutions to First Exam, Math 170, Section 002 Spring 2012

Multiple choice questions.

Question 1. You have 11 pairs of socks, 4 black, 5 white, and 2 blue, but they are not paired up. Instead, they are all mixed up in a drawer. It's early in the morning, and you don't want to turn on the lights in your dark room. How many socks must you pull out to guarantee that you have a pair of black socks?

- (a) 4 (b) 7 (c) 16
(d) 12 (e) 8 (f) 10

Answer 1. There are 8 black socks in the drawer and 14 socks which are either white or blue. If we are extremely unlucky it is possible to pull out 14 non-black socks. Thus to make sure we have at least 2 black socks at the end we must pull out at least 16 socks. The correct choice is (c). \square

Question 2. Consider the collection of all strings of length 11 composed of the digits 0 and 1 and such that there are no 1's next to each other. How many such strings are there?

- (a) 70 (b) 677 (c) 111
(d) 233 (e) 89 (f) 55

Answer 2. Let S_n be the number of such strings of length n . If $n = 1$, then we have two strings: the string [0] and the string [1]. If $n = 2$, then we have three strings like that: [00], [01], and [10]. Thus $S_1 = 2$ and $S_2 = 3$.

Suppose now we have such a string of length $n > 2$. If the string begins with 0 then the remaining $n - 1$ digits will form an arbitrary string of the same type. So, there will be S_{n-1}

such strings. If the string begins with 1, then the second digit must be 0, and the remaining $n - 2$ will form an arbitrary string of the same type. So, there will be S_{n-2} such strings.

All in all we will have that the total number of strings of length n is $S_n = S_{n-1} + S_{n-2}$. In other words the number of strings is a Fibonacci number. We compute:

$$S_1 = 2, S_2 = 3, S_3 = 5, S_4 = 8, S_5 = 13, S_6 = 21, \\ S_7 = 34, S_8 = 55, S_9 = 89, S_{10} = 144, S_{11} = 233.$$

The correct answer is (c). □

Question 3. Decompose the number 212 as a sum of non-consecutive Fibonacci numbers. What is the smallest such number?

- (a) 5 (b) 2 (c) 1
(d) 21 (e) 13 (f) 8

Answer 3. The biggest Fibonacci number that is less than 212 is 144. So $212 = 144 + 68$. The biggest Fibonacci number that is less than 68 is 55 so $212 = 144 + 55 + 13$. Since these are all Fibonacci numbers, the smallest number in the decomposition is 13. The correct answer is (c). □

Question 4. A family has six kids whose ages are all prime numbers. The older kids are 2, 6, 8, 12, and 18 years older than the youngest one. Which of the following is the age of the kid that is 8 year older than the youngest one?

- (a) 13 (b) 11 (c) 17
(d) 29 (e) 23 (f) 7

Answer 4. The age that we are looking for should be a prime number, such that 8 less than this number is also prime. Out of the six answers only 13 and 11 have this property. For each of these numbers we can compute the corresponding sequences of six ages:

- If the age we are looking for is 13, then the youngest kid is $13 - 8 = 5$ years old, and so the six ages are 5, $5 + 2 = 7$, $5 + 6 = 11$, $5 + 8 = 13$, $5 + 12 = 17$, $5 + 18 = 23$.

- If the age we are looking for is 11, then the youngest kid is $11 - 8 = 3$ years old, then the six ages are 3, $3 + 2 = 5$, $3 + 6 = 9$, $3 + 8 = 11$, $3 + 12 = 15$, and $3 + 18 = 21$.

Since only the first sequence of six numbers consists entirely of primes, the correct answer must be (a). \square

Question 5. What is the largest power of a prime that appears in the prime decomposition of 528.

- (a) 1 (b) 2 (c) 3
 (d) 4 (e) 5 (f) 6

Answer 5. First we compute the prime decomposition of 528. This number is even so we can divide it by 2: $528 = 2 \cdot 264 = 2 \cdot 2 \cdot 132 = 2 \cdot 2 \cdot 2 \cdot 66 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 33 = 2^4 \cdot 3 \cdot 11$. The correct answer is (d). \square

Question 6. What is the value of the product $3 \cdot 7 \cdot 11 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 113$ modulo 13?

- (a) 11 (b) 7 (c) 3
 (d) 8 (e) 5 (f) 9

Answer 6. To compute $3 \cdot 7 \cdot 11 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 113$ modulo 13 we proceed in stages.

$$3 \cdot 7 = 21 = -5 \pmod{13}.$$

Also

$$11 = -2 \pmod{13}, \quad 17 = 4 \pmod{13}, \quad \text{and} \quad 19 = 6 \pmod{13}.$$

This gives

$$3 \cdot 7 \cdot 11 \cdot 17 \cdot 19 = (-5) \cdot (-2) \cdot (4) \cdot 6 = 40 \cdot 6 \pmod{13}.$$

But

$$40 = 1 \pmod{13},$$

so

$$3 \cdot 7 \cdot 11 \cdot 17 \cdot 19 = 6 \pmod{13}.$$

Next we note that

$$23 = -3 \pmod{13}, \quad 29 = 3 \pmod{13}, \quad 113 = 9 = -4 \pmod{13}.$$

Then

$$23 \cdot 29 \cdot 113 = (-3) \cdot 3 \cdot (-4) = 3 \cdot 12 = 3 \cdot (-1) = -3 \pmod{13}.$$

Putting this together gives

$$3 \cdot 7 \cdot 11 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 113 = 6 \cdot (-3) = -18 = -5 = 8 \pmod{13}.$$

The correct choice is (d). □

Question 7. Find the missing digit in the UPC code

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(a) 1 (b) 4 (c) 2

(d) 0 (e) 5 (f) 9

Answer 7. Let x denote the missing digit. The checksum formula for this code gives

$$3 \cdot 0 + 2 + 3 \cdot x + 9 + 3 \cdot 0 + 0 + 3 \cdot 0 + 8 + 3 \cdot 5 + 4 + 3 \cdot 7 + 8 = 0 \pmod{10}.$$

Simplifying we get

$$3x = 3 \pmod{10}.$$

Thus $x = 1$ and the correct answer is (a). □

Question 8. Which of the following numbers are irrational?

(i) $\sqrt{2} + \frac{2}{3}$;

(ii) $\sqrt{\pi}$;

(iii) $3\sqrt{\frac{15}{60}}$;

(iv) $(2^{\sqrt{3}})^{\sqrt{3}}$.

(a) (i), (ii), and (iii) only (b) (ii) and (iii) only (c) (i) and (ii) only

(d) (iii) and (iv) only (e) (i) and (iv) only (f) (ii) and (iv) only

Answer 8. (i) If $Q := \sqrt{2} + \frac{2}{3}$ was rational, then $\sqrt{2} = Q - \frac{2}{3}$ will be rational, which will be a contradiction. Thus the number $\sqrt{2} + \frac{2}{3}$ is irrational.

(ii) If $Q := \sqrt{\pi}$ was rational, then $\pi = Q^2$ will be rational which is a contradiction. Thus $\sqrt{\pi}$ is irrational.

(iii) Note that the fraction $\frac{15}{60}$ reduces to $1/4$ and so

$$3\sqrt{\frac{15}{60}} = 3\sqrt{\frac{1}{4}} = 3\frac{1}{2} = \sqrt{3},$$

which is irrational. So $3\sqrt{\frac{15}{60}}$ is irrational.

(iv) We can simplify:

$$\left(2^{\sqrt{3}}\right)^{\sqrt{3}} = 2^{\sqrt{3} \cdot \sqrt{3}} = 2^3 = 8.$$

Therefore $\left(2^{\sqrt{3}}\right)^{\sqrt{3}}$ is rational.

The correct answer is (a). □

Question 9. True or False. Give a reason or a counter-example.

- (1) The set of all natural numbers that can be written with the digit 2 only, and the set of all natural numbers have the same cardinality.
- (2) Let A be the set of all even natural numbers ≥ 1 and ≤ 1000 , and let B be the set of all natural numbers ≥ 1 and ≤ 1000 that are divisible by 6. Then A and B have the same cardinality.
- (3) There are finitely many rational numbers between 0 and $\frac{1}{2}$.

(a)

(1)	(2)	(3)
T	T	T

(b)

(1)	(2)	(3)
T	F	T

(c)

(1)	(2)	(3)
F	T	F

(d)

(1)	(2)	(3)
F	F	F

(e)

(1)	(2)	(3)
T	F	F

(f)

(1)	(2)	(3)
T	T	F

Answer 9. (1) is True. We can construct a one-to-one correspondence

$$\left\{ \begin{array}{l} \text{all natural numbers} \\ \text{made out of the} \\ \text{digit 2} \end{array} \right\} \longleftrightarrow \{\text{all natural numbers}\}$$

explicitly by pairing:

$$\left[\underbrace{22 \cdots 2}_{n\text{-digits}} \right] \longleftrightarrow \left[\begin{array}{l} \text{the natural} \\ \text{number } n \end{array} \right].$$

(2) is False. The sets A and B are both finite sets of natural numbers. Moreover the set B is obtained from the set A by removing from A all numbers that are not divisible by 6. For instance we must remove 2, 4, 8, etc.. Since removing an element in a finite set decreases its cardinality, it follows that A and B can not have the same cardinality.

(3) is False. Indeed the numbers of the form $1/2^n$, $n = 1, 2, \dots$ are all between 0 and $1/2$ and the set of such numbers is in a one-to-one correspondence with all natural numbers by pairing $1/2^n$ with n .

The correct answer is (e). □

Question 10. Consider the following sets of real numbers:

- The set A of all negative integers;
- The set B of all real numbers between 0 and 1 having only 0's and 1's after the decimal point.
- The set C of all real numbers between 0 and 1 having only 1's and 2's after the decimal point.

Which of the following statements is correct?

(a) B and C have different cardinalities

(b) A and B have the same cardinality

(c) B has a smaller cardinality than C

(d) C has a smaller cardinality than B

(e) A and C have the same cardinality

(f) A has a smaller cardinality than B and also a smaller cardinality than C

Answer 10. The set A has the same cardinality as the set of natural numbers: pair a natural number n with the negative integer $-n$. The set B has a bigger cardinality. We can see this by Cantor's diagonalization argument. Indeed, suppose that we can find a one-to-one correspondence between the set B and all natural numbers. Let b_n denote the number in B that corresponds to the natural number n . We will construct a number a in B that is not equal to any one of the b_n 's. Define a to be the number between 0 and 1 for which:

$$\left[\begin{array}{l} n\text{-th digit of } a \text{ after the} \\ \text{decimal point} \end{array} \right] = \begin{cases} 0 & \text{if the } n\text{-th digit of } b_n \text{ after} \\ & \text{the decimal point is 1;} \\ 1 & \text{if the } n\text{-th digit of } b_n \text{ after} \\ & \text{the decimal point is 0.} \end{cases}$$

This defines a completely and ensures that a and b_n have a different n -th digit after the decimal place. Therefore a can not be equal to b_n for any n . Which is a contradiction since $a \in B$ and the b_n 's enumerated all the numbers in B .

This shows that B has a bigger cardinality than A . Similarly we can argue that C has a bigger cardinality than A . The correct answer is (f). \square