LOCATIONS OF MODULES FOR BRAUER TREE ALGEBRAS

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Abstract. Suppose Λ is a Brauer tree algebra. We determine the location of M in the stable Auslander-Reiten quiver of Λ from the description of M as a multi-pushout of elementary modules. This is done via a walk around the Brauer tree of Λ which is associated to M. As an application, we determine the group of stable homomorphisms between two nonprojective indecomposable Λ-modules from their multi-pushout descriptions.

1. Introduction

Suppose Λ is a Brauer tree algebra over an algebraically closed field k with Brauer tree $T(\Lambda)$. Let $\mathrm{Ind}(\Lambda)$ be the finite set of isomorphism classes $[M]$ of finitely generated nonprojective indecomposable Λ-modules M. There are two different ways of paramaterizing the elements of $\mathrm{Ind}(\Lambda)$. The first is to use the description of M as a multi-pushout of elementary modules, as given by the work of Janusz and Kupisch [11, 12]. The second results from the fact that $\mathrm{Ind}(\Lambda)$ is the set of vertices of the stable Auslander-Reiten quiver $\Gamma_s(\Lambda)$ of Λ (see [2, 3]). The main results of this paper, Theorem 2.6 and Corollary 2.9, provide an explicit comparison between these two parameterizations. This is done by determining for all $[M], [N] \in \mathrm{Ind}(\Lambda)$, both the location of $[M]$ and the relative locations of $[M]$ and $[N]$ in $\Gamma_s(\Lambda)$ from the multi-pushout descriptions of M and N, using certain walks around $T(\Lambda)$ which are associated to M and N.

The motivation for Theorem 2.6 and Corollary 2.9 is that the two above parameterizations of $\mathrm{Ind}(\Lambda)$ are useful for different reasons. Multi-pushouts provide a way to specify concrete modules. In particular, one can describe M as a multi-pushout (thus fixing $[M]$) by giving a certain "top-socle path" in $T(\Lambda)$ associated to M and an integer pertaining to the composition factors of M which are adjacent to the exceptional vertex of $T(\Lambda)$ (see Definition 2.1). The stable Auslander-Reiten quiver $\Gamma_s(\Lambda)$ has other applications. One due to Gabriel and Riedtmann [9] is that from the locations of two vertices $[M]$ and $[N]$ in $\Gamma_s(\Lambda)$, one can find the group of stable homomorphisms $\underline{\mathrm{Hom}}_\Lambda(M,N)$. In particular, one can find $\dim_k \underline{\mathrm{Hom}}_\Lambda(M,N)$ by analyzing paths in $\Gamma_s(\Lambda)$ from $[M]$ to $[N]$.

Theorem 2.6 and Corollary 2.9 have the following form. It is known that $\Gamma_s(\Lambda)$ is a finite graph which is isomorphic to a tube in the sense of [3, Thm. 6.5.5]. To find the location of $[M]$ in $\Gamma_s(\Lambda)$ up to a graph automorphism of $\Gamma_s(\Lambda)$, it suffices to find the minimal distance from $[M]$ to one of the boundaries of $\Gamma_s(\Lambda)$. We determine this distance from a particular walk around $T(\Lambda)$ associated to the multi-pushout description of M. An analysis of the distance between modules on

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the same boundary of $\Gamma_s(\Lambda)$ then leads to an analogous method of finding the relative locations of any two vertices $[M]$ and $[N]$ in $\Gamma_s(\Lambda)$ from walks around $T(\Lambda)$.

In Section 4 we give an application of Theorem 2.6 and Corollary 2.9 to determining $\underline{\mathrm{Hom}}_\Lambda(M,N)$ from the multi-pushout descriptions of M and N. This is done by first finding the locations of $[M]$ and $[N]$ in $\Gamma_s(\Lambda)$, and by then applying the results of Gabriel and Riedtmann in [9]. We should point out that in principle Gabriel's and Riedtmann's work in [9] can also be used to find $\underline{\mathrm{Hom}}_\Lambda(M,N)$ from the multi-pushout descriptions of M and N. Their approach involves first constructing a rather explicit stable equivalence between Λ and a serial Brauer tree algebra Λ'; they then treat the serial case in detail. The difference between our method and theirs is thus that we use computations that only involve walks around $T(\Lambda)$ rather than a computation of the images of M and N under a stable equivalence between Λ and a serial algebra. For some other descriptions of $\underline{\mathrm{Hom}}_\Lambda(M,N)$, see [5], [7] and their references.

A case of particular interest is when $M = N$. Then $\underline{\mathrm{Hom}}_\Lambda(M,N)$ is the stable endomorphism ring $\underline{\mathrm{End}}_\Lambda(M)$ of M. It was shown by Gabriel and Riedtmann that $\underline{\mathrm{End}}_\Lambda(M)$ is generated over k by a single nilpotent element. Our methods lead in Theorem 3.2 to the determination of an explicit nilpotent generator, which we call the "shift" of M, and whose nilpotency we compute via the top-socle path of M. These results do not use Theorem 2.6 or Corollary 2.9, but can be deduced directly from the multi-pushout description of M. Theorem 3.2 is straightforward in case Λ is a serial algebra, so this is not the case of primary interest. The methods of Gabriel and Riedtmann give in principle a different way (via the stable equivalence discussed above) to determine an explicit nilpotent generator for $\underline{\mathrm{End}}_\Lambda(M)$. However, it is not clear that their approach would lead to a shorter proof of Theorem 3.2.

Suppose Λ is a block with cyclic defect groups of the group ring kG of a finite group G. This paper arose in part from the arithmetic interest of computing the universal deformation ring of a Λ-module M from the multi-pushout description of M. It is known (see [14],[4]) that an indecomposable Λ-module M has a universal deformation ring if $\underline{\mathrm{End}}_\Lambda(M) = k$. Whether $\underline{\mathrm{End}}_\Lambda(M) = k$ can be determined from the top-socle path of M via Theorem 3.2. It is shown in [4] that when $\underline{\mathrm{End}}_\Lambda(M) = k$, the universal deformation ring of M may then be determined from $\dim_k \mathrm{Ext}^i_\Lambda(M,M)$ for $i = 1$ and 2. Since $\mathrm{Ext}^i_\Lambda(M,M) = \underline{\mathrm{Hom}}_\Lambda(\Omega^i(M),M)$, where Ω is the Heller operator, the dimensions of these Ext groups may be determined via Theorem 2.6 and Corollary 2.9, as shown in Section 4.

We end this introduction with a summary of the contents of this paper. In Section 2 we prove our main results concerning the location of $[M]$ and the relative locations of $[M]$ and $[N]$ in $\Gamma_s(\Lambda)$. In Section 3 we prove Theorem 3.2 concerning $\underline{\mathrm{End}}_\Lambda(M)$. In Section 4 we apply the results of Sections 2 and 3 to the study of $\underline{\mathrm{Hom}}_\Lambda(M,N)$ and $\mathrm{Ext}^i_\Lambda(M,N)$ for $i > 0$.

Throughout this paper, k is an algebraically closed field and Λ is an arbitrary Brauer tree algebra over k with Brauer tree $T(\Lambda)$, multiplicity $m \geq 1$ and e isomorphism classes of simple modules. For the definition of a Brauer tree algebra we refer to [1, Section 17]. We also need some basic results from Auslander-Reiten theory as may be obtained from [2]. The almost split sequences for Brauer tree algebras have been first discussed in [15]. Since Brauer tree algebras are especially string algebras, we will also use the notation introduced in [6, Section 3]. Note that we compose morphisms as if they were written on the right, so that fg is the composition of $f : X \to Y$ with $g : Y \to Z$.

2. The location of an indecomposable module in $\Gamma_s(\Lambda)$

In this section we start with an arbitrary nonprojective indecomposable Λ-module M given as a multi-pushout of elementary modules (see [11, 12] and [15]). We determine the location of $[M]$ in the stable Auslander-Reiten quiver $\Gamma_s(\Lambda)$, using certain walks around $T(\Lambda)$. We assume that $\Gamma_s(\Lambda)$ contains more than one module. In the following we will not distinguish between the module M and its isomorphism class $[M]$.

Recall that a module is called elementary if it is a proper factor module of a projective indecomposable module. This means that M can be written in the following way

$$M = \begin{array}{ccccccccccccc} & U_1 & & & & U_2 & & & & & & U_t & \\ M_{1,1} & & V_{1,2} & & V_{2,1} & & V_{2,2} & & \cdots & & V_{t,1} & & M_{t,2} \\ & & & T_1 & & & & T_2 & & T_{t-1} & & & \end{array}$$

Here the U_i and T_j are simple modules, and there exist elementary modules X_i of the form $X_i = \begin{array}{ccc} & U_i & \\ M_{i,1} & & M_{i,2} \end{array}$ such that $V_{i,j} = M_{i,j}/\mathrm{soc}(M_{i,j})$ and $\mathrm{soc}(M_{i,2}) \cong \mathrm{soc}(M_{i+1,1}) \cong T_i$. Furthermore, $\begin{array}{ccc} U_i & & U_{i+1} \\ V_{i,2} & & V_{i+1,1} \\ & T_i & \end{array}$ is a submodule of the projective cover P_{T_i} of T_i.

Note that the mirror image $\tilde{M}$ of M

$$\tilde{M} = \begin{array}{ccccccccccccc} & U_t & & & & & & U_2 & & & & U_1 & \\ M_{t,2} & & V_{t,1} & & \cdots & & V_{2,2} & & V_{2,1} & & V_{1,2} & & M_{1,1} \\ & & & T_{t-1} & & T_2 & & & & T_1 & & & \end{array}$$

is isomorphic to M.

Let $T_0 = \mathrm{soc}(M_{1,1})$ if $M_{1,1} \neq 0$ and $T_t = \mathrm{soc}(M_{t,2})$ if $M_{t,2} \neq 0$. The sequence of edges in $T(\Lambda)$ that corresponds to the sequence of simple modules in the top and socle of M

$$T_0 \text{ (if } M_{1,1} \neq 0), U_1, T_1, U_2, \ldots, T_{t-1}, U_t, T_t \text{ (if } M_{t,2} \neq 0)$$

is a path in $T(\Lambda)$ [15, p. 127]. We call this sequence the *top-socle path* of M and denote it by $E_1, \ldots, E_s$. If none of the E_j is adjacent to the exceptional vertex, then all E_j are distinct. Otherwise two consecutive edges E_i, E_{i+1} which are adjacent to the exceptional vertex are equal. But then none of the other E_j is adjacent to the exceptional vertex, and the U_y, respectively T_z, are pairwise distinct. So $\mathrm{top}(M)$ and $\mathrm{soc}(M)$ have no repeated composition factors.

We want to find the distance from M to one of the boundaries of the stable Auslander-Reiten quiver and thus determine the location of M in $\Gamma_s(\Lambda)$. In other words, we determine the length of a maximal directed path starting at M. Note that we call a path in $\Gamma_s(\Lambda)$ *directed* if it does not contain any subpath from $\Omega^2(X)$ to X for any nonprojective indecomposable Λ-module X. Further, a directed path is called *maximal directed* if it ends at one of the boundaries of $\Gamma_s(\Lambda)$.

To describe the modules along a path in $\Gamma_s(\Lambda)$ we use the notation introduced in [6, Section 3]. This is possible, since Λ is in particular a string algebra. Additionally, we call a uniserial module H a *hook* if $H = \begin{matrix} S \\ W \end{matrix}$, where S is simple and the corresponding projective cover P_S has the form $P_S = \begin{matrix} & S & \\ W & & Q \\ & S & \end{matrix}$. On the other hand, a uniserial module C is called a *cohook* if $C = \begin{matrix} X \\ R \end{matrix}$, where R is simple and the corresponding projective cover P_R has the form $P_R = \begin{matrix} & R & \\ X & & Q' \\ & R & \end{matrix}$. Note that W or X might be zero. So there are exactly twice as many hooks as isomorphism classes of simple Λ-modules, and the hooks are exactly the modules lying at the boundaries of $\Gamma_s(\Lambda)$. If $e > 1$ then a hook H is uniquely determined by $\mathrm{soc}(H)$ and $\mathrm{top}(H)$. If $e = 1$ then there are exactly two hooks, namely the unique simple module S and P_S/S. Since Λ is a Brauer tree algebra, every cohook is a hook and vice versa. So the module M_h (respectively ${}_hM$), as described in [6, Section 3], can be obtained from M by adding a hook on the right end (respectively left end) of M. On the other hand, the module M_c (respectively ${}_cM$) can be obtained from M by removing a cohook on the right end (respectively left end) of M.

We will show that one of the two maximal directed paths in $\Gamma_s(\Lambda)$ starting at M can be obtained as follows. Starting with M we either add a hook on the right end or remove a cohook on the right end of each successive module we obtain. After a certain number of steps we come to a hook which belongs to one of the boundaries of $\Gamma_s(\Lambda)$. We call this maximal directed path starting at M the *maximal directed right-oriented path* starting at M. This path can be encoded by writing down the sequence of the simple Λ-modules occurring as rightmost composition factor of each module on the path. Since each simple Λ-module corresponds to an edge in $T(\Lambda)$, it is a natural question how this sequence of rightmost composition factors can be described using only $T(\Lambda)$. It turns out that this sequence is obtained by taking a clockwise "walk" around $T(\Lambda)$ from the rightmost composition factor of M to the rightmost composition factor of the target hook, and then deleting every other edge occurring in this walk. If the multiplicity m of $T(\Lambda)$ is 1, then we take the shortest such clockwise walk. If $m > 1$ then it can happen that we have to walk around $T(\Lambda)$ several times before we have met all the edges that occur in the sequence of rightmost composition factors. Here the number of times we have to walk around $T(\Lambda)$ depends on the maximal number of times an edge adjacent to the exceptional vertex is a composition factor of M. Having obtained the sequence of rightmost composition factors in this way, we then only have to count the number of edges (with multiplicities) appearing in this sequence to get the distance from M to one of the boundaries.

Note that we can get the other maximal directed path in $\Gamma_s(\Lambda)$ starting at M by successively adding a hook or removing a cohook on the left end of the modules. In the following we will give a formula for the distance from M to one of the boundaries of $\Gamma_s(\Lambda)$, using the principle of finding the walk associated to the sequence of the rightmost composition factors of the modules occurring on the maximal directed right-oriented path starting at M.

In order to describe this distance we need another description of M as follows.

Definition 2.1. Let $E_1, \dots, E_s$ be the top-socle path of M. Let $\epsilon = (\epsilon_1, \epsilon_s)$ be given as $\epsilon = (-1, 1)$ if M is simple. Otherwise let $\epsilon_1 = -1$ if E_1 belongs to $\mathrm{soc}(M)$, and let $\epsilon_1 = 1$ if E_1 belongs to $\mathrm{top}(M)$. Similarly let $\epsilon_s = -1$ if E_s belongs to $\mathrm{soc}(M)$, and let $\epsilon_s = 1$ if E_s belongs to $\mathrm{top}(M)$. Note that ϵ_s is determined by ϵ_1 and s. In case that the multiplicity of $T(\Lambda)$ is $m > 1$ and one of the edges E_{i_0} is adjacent to the exceptional vertex, let μ be equal to the number of times E_{i_0} occurs as a composition factor of M. Note that μ is independent of the choice of the edge E_{i_0} adjacent to the exceptional vertex. If no edge in $E_1, \dots, E_s$ is adjacent to the exceptional vertex, let $\mu = 0$.

Then $E_1, \dots, E_s$, ϵ and μ uniquely determine M. In the following we call ϵ the *direction of* M, and μ is called the *multiplicity of* M.

Definition 2.2. A *clockwise walk* around the Brauer tree $T(\Lambda)$ is a finite sequence of edges $X_1, \dots, X_n$ and of vertices $v_1, \dots, v_{n+1}$ of $T(\Lambda)$, written

$$W = (v_1, X_1, v_2, \dots, v_n, X_n, v_{n+1}),$$

where $n \geq 0$, such that

(i) $v_i \neq v_{i+1}$ are the end points of X_i.

(ii) X_{i+1} is the edge that is the next clockwise edge to X_i around the vertex v_{i+1}. If v_{i+1} is a leaf vertex, meaning X_i is the only edge adjacent to v_{i+1}, then $X_{i+1} = X_i$.

The walk W is called a *complete walk of multiplicity* ω if $n = 2e\omega$, where e denotes as before the number of nonisomorphic simple Λ-modules. Note that in that case we walk around $T(\Lambda)$ exactly ω times.

We also need the following definition to be able to describe the distance from M to the boundaries of $\Gamma_s(\Lambda)$. This definition gives the starting and the end point of the walk around $T(\Lambda)$ that corresponds to the rightmost composition factors of the modules occurring on the maximal directed right-oriented path in $\Gamma_s(\Lambda)$ starting at M.

Definition 2.3. Let M be a nonprojective indecomposable Λ-module with top-socle path $E_1, \dots, E_s$, direction ϵ and multiplicity μ. If M is a hook, we assume, by taking the mirror image of M if necessary, that $\epsilon = (-1, 1)$.

(i) If M is a hook then let v_a be one endpoint of E_s and let v_z be the other endpoint of E_s such that v_z is nonexceptional.

Now suppose that M is not a hook.

(ii) If $s = 1$, i.e. M is simple, then let v_a be one endpoint of M and let v_z be the other endpoint of M.

Let now $s \geq 2$ and let M be not a hook. (Case (iii) is illustrated below.)

(iii) If $\epsilon_s = 1$, i.e. E_s belongs to $\mathrm{top}(M)$, then let v_a be the endpoint of E_s which is not an endpoint of E_{s-1}, if $E_s \neq E_{s-1}$. If $E_s = E_{s-1}$ then E_s is adjacent to the exceptional vertex, and we let v_a be the endpoint of E_s which is not the exceptional vertex. If $\epsilon_s = -1$, i.e. E_s belongs to $\mathrm{soc}(M)$, then let v_a be the common vertex of E_s and E_{s-1}, if $E_s \neq E_{s-1}$. If $E_s = E_{s-1}$ then E_s is adjacent to the exceptional vertex, and we let v_a be the exceptional vertex.

If $\epsilon_1 = 1$, i.e. E_1 belongs to $\mathrm{top}(M)$, then let v_z be the endpoint of E_1 which is not an endpoint of E_2, if $E_1 \neq E_2$. If $E_1 = E_2$ then E_1 is adjacent to the exceptional vertex, and we let v_z be the endpoint of E_1 which is not the exceptional vertex. If $\epsilon_1 = -1$, i.e. E_1 belongs to $\mathrm{soc}(M)$, then let v_z be the common vertex of E_1 and E_2, if $E_1 \neq E_2$. If $E_1 = E_2$ then E_1 is adjacent to the exceptional vertex, and we let v_z be the exceptional vertex.

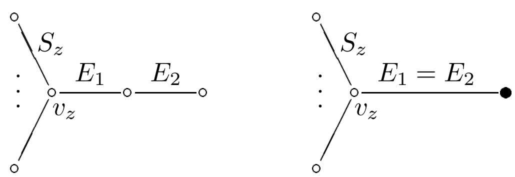
In case (i), let $S_a = S_z = E_s$. In case (ii) or (iii), let $S_a = E_s$, and let S_z be the next counterclockwise edge to E_1 around v_z. If v_z is a leaf then let $S_z = E_1$.

We want to illustrate case (iii) of this definition. Here $\bullet$ denotes the exceptional vertex.

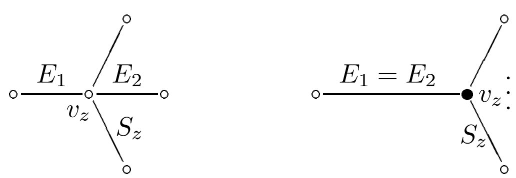
$\epsilon_s = 1$:

$\epsilon_s = -1$:

$\epsilon_1 = 1$:



$\epsilon_1 = -1$:



The following two theorems describe the distance from M to the boundaries of $\Gamma_s(\Lambda)$, and thus the location of M in $\Gamma_s(\Lambda)$. We first describe the easier case when the multiplicity of $T(\Lambda)$ is $m = 1$.

Theorem 2.4. *Suppose the multiplicity of $T(\Lambda)$ is $m = 1$. Let M be a nonprojective indecomposable Λ-module with ϵ, μ, and v_a, v_z, S_a, S_z given as in Definition 2.3.*

Let $W_M = (v_1, X_1, \ldots, X_n, v_{n+1})$ be the shortest clockwise walk around $T(\Lambda)$ with $v_1 = v_a$, $X_1 = S_a$ and $X_n = S_z$, $v_{n+1} = v_z$. Let H be the unique hook with $\mathrm{soc}(H) = E_1$ and $\mathrm{top}(H) = X_n$.

There exists a unique directed path in $\Gamma_s(\Lambda)$ starting at M and ending at H which is obtained by successively adding hooks or removing cohooks on the right end of the modules and for which the sequence C_M of rightmost composition factors is given by

$$C_M = (X_1, X_3, X_5, \ldots, X_n).$$

In particular, n is odd. The distance d from M to the boundary of $\Gamma_s(\Lambda)$ containing H is given by $d = (n-1)/2$. The distance from M to the other boundary is given by $e - 1 - d$. Thus the location of M in $\Gamma_s(\Lambda)$ is controlled by the walk W_M.

Theorem 2.4 is a corollary of Theorem 2.6 which gives the distance from M to one of the boundaries of $\Gamma_s(\Lambda)$ in the general case, which means that m is arbitrary. In this case the definition of the clockwise walk around $T(\Lambda)$ corresponding to the sequence C_M of rightmost composition factors is more complicated. It consists of a certain shortest clockwise walk W_M which meets all the edges in the top-socle path of M and a complete walk of a certain multiplicity η which depends on the multiplicity μ of M. Since W_M and η are only important to understand the technical details of Theorem 2.6, we define them in a separate definition before we state the theorem.

We use the convention that if $(b_1, \dots, b_x)$ and $(c_1, \dots, c_y)$ are two sequences, then $(b_1, \dots, b_x)(c_1, \dots, c_y)$ denotes the sequence $(b_1, \dots, b_x, c_1, \dots, c_y)$.

Definition 2.5. Let M be a nonprojective indecomposable Λ-module with top-socle path $E_1, \dots, E_s$, direction ϵ and multiplicity μ. If M is a hook, we assume that $\epsilon = (-1, 1)$. Let v_a, v_z, S_a, S_z be given as in Definition 2.3.

If $s = 1$ or $s = 2$, let A be the empty sequence. Otherwise, if $\epsilon_1 = 1$, let A be the sequence $A = (E_s, E_{s-1}, \dots, E_1)$. If $\epsilon_1 = -1$, let $A = (E_s, E_{s-1}, \dots, E_2)$. If A is not the empty sequence, then $A = (A_1, A_2, \dots, A_t)$ where $t = s$ or $s - 1$.

Let $W_M = (v_1, X_1, v_2, X_2, \dots, X_n, v_{n+1})$ be the shortest clockwise walk around $T(\Lambda)$ with $v_1 = v_a$, $X_1 = S_a$ and $X_n = S_z$, $v_{n+1} = v_z$, such that, if A is not the empty sequence, there exist $i_1 < i_2 < \cdots < i_t$ with $X_{i_1} = A_1, X_{i_2} = A_2, \dots, X_{i_t} = A_t$. Note that W_M can contain a complete walk of multiplicity 1.

Let $W_o = (w_1, Y_1, w_2, Y_2, \dots, Y_{2e}, w_{2e+1})$ be the complete walk of multiplicity 1 such that $W_M W_o$ is a clockwise walk around $T(\Lambda)$.

Let η be the integer attached to M which is given as follows:

(i) If none of the vertices v_i in W_M is the exceptional vertex, then $\eta = 0$.
(ii) If one of the v_i is the exceptional vertex, but $\mu = 0$, then $\eta = m - 1$.

If $\mu \geq 1$ and one of the v_i is the exceptional vertex, then let j_0 be the smallest index such that E_{j_0} is adjacent to the exceptional vertex.

(iii) If $s = 1$, i.e. M is simple, then $\eta = \mu - 1 = 0$ if v_a is exceptional and v_z is nonexceptional. Otherwise v_a is nonexceptional and v_z is exceptional, and $\eta = m - \mu = m - 1$.

Let now M be not simple.

(iv) If E_{j_0} belongs to $\operatorname{soc}(M)$, then $\eta = m - \mu$.
(v) If E_{j_0} belongs to $\operatorname{top}(M)$ and E_{j_0+1} exists and is not adjacent to the exceptional vertex, then $\eta = m - \mu$. In this case j_0 must be 1 and $\mu = 1$.
(vi) If E_{j_0} belongs to $\operatorname{top}(M)$ and E_{j_0+1} exists and is equal to E_{j_0}, then $\eta = \mu - 2$.
(vii) If E_{j_0} belongs to $\operatorname{top}(M)$ and either $j_0 = s$ or E_{j_0+1} is adjacent to the exceptional vertex and not equal to E_{j_0}, then $\eta = \mu - 1$.

Theorem 2.6. *Let M be a nonprojective indecomposable Λ-module with $E_1, \dots, E_s$, ϵ, μ, and $W_M = (v_1, X_1, \dots, X_n, v_{n+1})$, $W_o = (w_1, Y_1, \dots, Y_{2e}, w_{2e+1})$, η given as in Definition 2.5. If $e > 1$, let H be the unique hook with $\operatorname{soc}(H) = E_1$ and $\operatorname{top}(H) = X_n$. In case $e = 1$, let H be the unique simple module if v_1 is exceptional, otherwise let H be the hook with multiplicity m.*

There exists a unique directed path in $\Gamma_s(\Lambda)$ starting at M and ending at H which is obtained by successively adding hooks or removing cohooks on the right end of the modules and for which the sequence C_M of rightmost composition factors is given by

$$C_M = (X_1, X_3, \dots, X_n)(Y_2, Y_4, \dots, Y_{2e})^\eta.$$

In particular, n is odd and $Y_{2e} = X_n$. The distance d from M to the boundary of $\Gamma_s(\Lambda)$ containing H is given by

$$d = \text{length}(C_M) - 1 = (n-1)/2 + \eta e.$$

The distance from M to the other boundary is given by $me - 1 - d$, since the length of any longest directed path which connects the two boundaries is exactly $me - 1$. Thus the location of M in $\Gamma_s(\Lambda)$ is controlled by the walk W_M and the integer η.

Remark 2.7. (i) Theorem 2.4 follows from Theorem 2.6 in the following way. Let the multiplicity of $T(\Lambda)$ be $m = 1$ and let M be an arbitrary nonprojective indecomposable Λ-module. Then the top-socle path of M, and thus the sequence A of Definition 2.5, contains no multiple edges. Further, the shortest clockwise walk $W_M = (v_1, X_1, \dots, X_n, v_{n+1})$ with $v_1 = v_a$, $X_1 = S_a$ and $X_n = S_z$, $v_{n+1} = v_z$ meets automatically all the edges in the sequence A, since A does not contain any multiple edges. Since we are in case (i) of Definition 2.5, it follows that $C_M = (X_1, X_3, \dots, X_n)$ and $d = (n-1)/2$. This is exactly the description of C_M and d in Theorem 2.4.

(ii) In case the multiplicity m of $T(\Lambda)$ is arbitrary and M is a nonprojective indecomposable Λ-module of multiplicity $\mu = 0$, W_M can be determined as in Theorem 2.4. This follows since we are then either in case (i) or in case (ii) of Definition 2.5, and thus C_M is either $C_M = (X_1, X_3, \dots, X_n)$ or $C_M = (X_1, X_3, \dots, X_n)(Y_2, Y_4, \dots, Y_{2e})^{m-1}$. The distance d is then either $d = (n-1)/2$ or $d = (n-1)/2 + (m-1)e$.

Proof. Let first M be a hook, which means that M belongs to one of the boundaries of $\Gamma_s(\Lambda)$. Then M is uniserial with $\epsilon = (-1, 1)$, and either $\mu = 0$, or $\mu = 1 < m$, or $\mu = m > 1$. By Definitions 2.3 and 2.5, $S_a = S_z = E_s$ and $W_M = (v_a, E_s, v_z)$ such that v_z is nonexceptional. If $\mu = 0$ then also v_a is nonexceptional and we are in case (i) of Definition 2.5, which means that $\eta = 0$. If $\mu = 1 < m$ then either v_a is exceptional or not. If v_a is exceptional and M is simple, then we are in case (iii) and $\eta = \mu - 1 = 0$. If v_a is exceptional and M is not simple, then, since $\mu = 1$, E_1 is not adjacent to the exceptional vertex and we are in case (vii), which implies that $\eta = \mu - 1 = 0$. If v_a is nonexceptional, then we are in case (i) and again $\eta = 0$. If $m > 1$ and $\mu = m$, then all composition factors of M are edges which are adjacent to the exceptional vertex, which means that we are in case (iv) and it follows that $\eta = m - \mu = 0$. Thus in all cases $\eta = 0$ and so, according to Theorem 2.6, $C_M = (E_s)$, and $d = 0$. Since M does lie at distance 0 from one of the boundaries of $\Gamma_s(\Lambda)$, this establishes Theorem 2.6 if M is a hook.

Let now M be not a hook and let $W_M = (v_1, X_1, \dots, X_n, v_{n+1})$ be the corresponding clockwise walk around $T(\Lambda)$, as in Definition 2.5. We want to use an inductive argument to verify the description of the sequence C_M and thus of the distance d.

Define a path Θ_M in $\Gamma_s(\Lambda)$ by starting with M and then either adding a hook on the right end or removing a cohook on the right end of each successive module. Let

N be a module on Θ_M. Then the successor N' of N is well-defined by this procedure unless N is simple and not a hook. In the latter case, if $N = M$ we require N' to be obtained by adding to M the hook whose top is the simple module which is the next clockwise edge to M around v_z. If $N \neq M$ we require N' to be the neighbor of N which is not isomorphic to $\Omega^{-2}(N'')$, where N'' is the predecessor of N on Θ_M. From this requirement and from the description of the almost split sequences, it follows that there are no three consecutive modules of the form $\Omega^2(R), R', R$ on the path Θ_M. This means that Θ_M is a directed path starting at M, and so it must reach one of the boundaries of $\Gamma_s(\Lambda)$ and thus a hook, since $\Gamma_s(\Lambda)$ is a finite tube.

Let now M' be the indecomposable Λ-module which is a neighbor of M in $\Gamma_s(\Lambda)$ and which is given by either $M = M'_c$ if M starts on a peak, or otherwise by $M' = M_h$. Since M' is obtained from M by either adding a hook on the right end or by removing a cohook on the right end of M, it follows that M' lies on the path Θ_M starting at M. In fact, M' is the next module after M on Θ_M. In particular, this shows that by successively either adding a hook or removing a cohook on the right end, we move from M and from M' to the same hook H belonging to one of the boundaries of $\Gamma_s(\Lambda)$.

Since the path Θ_M is directed, it describes the unique minimal path from M to H. Further, Θ_M is the maximal directed right-oriented path in $\Gamma_s(\Lambda)$ starting at M. Thus the distance d from M to one of the boundaries of $\Gamma_s(\Lambda)$ is given by the number of modules in this minimal path from M to H minus 1.

We assume now that the sequence of the rightmost composition factors of the modules on the minimal path from M' to H in $\Gamma_s(\Lambda)$ is given by $C_{M'}$, where $C_{M'}$ is the sequence obtained by applying Theorem 2.6 to M'. We want to show that then the sequence C_M, as given in Theorem 2.6, satisfies $C_M = (E_s)C_{M'}$. This then implies that C_M is the sequence of the rightmost composition factors of the modules on the minimal path from M to H in $\Gamma_s(\Lambda)$. Further, it follows that d, as given in Theorem 2.6, gives the distance from M to H, which is the distance from M to one of the boundaries of $\Gamma_s(\Lambda)$. Note that this also implies that H is the unique hook with $\mathrm{soc}(H) = E_1$ and $\mathrm{top}(H) = X_n$ if $e > 1$. In case $e = 1$, it follows that H is simple if v_1 is exceptional, otherwise H has multiplicity m.

Let now μ' be the multiplicity of M', and let η' be the integer attached to M' according to Definition 2.5. To show $C_M = (E_s)C_{M'}$, we have to look at two different cases.

1. $M = M'_c$. Then either

$$\text{(a) } M = \begin{array}{cccc} & & & E_s \\ M' & & U & \\ & E_{s-1} & & \end{array}$$

where U is uniserial and $\begin{array}{c} E_s \\ U \\ E_{s-1} \end{array}$ is a cohook, or

$$\text{(b) } M = \begin{array}{cccc} & E_{s-1} & & \\ N & & V & \\ & & & E_s \end{array} = \begin{array}{cc} M' & \\ & E_s \end{array}$$

where V is uniserial and one endpoint of E_s is a nonexceptional leaf vertex. Note that this leaf vertex cannot be exceptional, since otherwise E_s is not a cohook.

In case (a), the walk for M is

$$W_M = (v_1, E_s, v_2, E_{s-1}, v_3, X_3, \ldots, X_n, v_{n+1}).$$

In case (b), this walk is

$$W_M = (v_1, E_s, v_2, E_s, v_3, X_3, \ldots, X_n, v_{n+1}).$$

In both cases $n \geq 3$ because M is not a hook. Since X_3 is the rightmost composition factor of M', the walk for M' is given by

$$W_{M'} = (v_3, X_3, \ldots, X_n, v_{n+1}) W_o^{\gamma},$$

where W_o is the complete walk around $T(\Lambda)$ of multiplicity 1 as given in Definition 2.5, and γ is either 0 or 1. In particular, W_o^0 is the (empty) walk of length 0.

The top-socle path of M' has either the form $E_1, \ldots, E_{s'}, X_3$ or the form $E_1, \ldots, E_{s'}$, where $s' = s - 2$ in case (a) and $s' = s - 1$ in case (b). In both cases $E_{s'}$ belongs to $\mathrm{top}(M')$ and $\mathrm{top}(M)$. The latter case $E_1, \ldots, E_{s'}$ occurs if the uniserial part of M having $E_{s'}$ in the top and $E_{s'+1}$ in the socle consists only of $E_{s'}, E_{s'+1}$. In that case $X_3 = E_{s'}$.

The complete walk W_o occurs in $W_{M'}$ if and only if $(v_3, X_3, \ldots, X_n, v_{n+1})$ does not contain the subsequence A' associated to M' as in Definition 2.5. By comparing M and M' in the various cases listed above, we see that W_o occurs in $W_{M'}$ if and only if the top-socle path of M' is $E_1, \ldots, E_{s'}, X_3$ and $(v_3, X_3, \ldots, X_n, v_{n+1})$ does not meet $E_{s'}$ after meeting X_3 as first edge. This happens exactly if $E_{s'} = X_3$ and $E_{s'}$ is adjacent to the exceptional vertex. Note that in this case the multiplicities μ and μ' are the same if and only if $E_{s'+1} \neq E_{s'}$.

If the multiplicities μ and μ' of M and M', respectively, are the same, and W_o does not occur, i.e. $\gamma = 0$, we only have to show that $\eta = \eta'$. If we are in case (i) (respectively case (ii)) of Definition 2.5 for M, then v_1 and v_2 are both nonexceptional, so we are in case (i) (respectively case (ii)) for M', and so $\eta = \eta'$. Since M is not simple, case (iii) does not occur. If we are in case (iv) for M, then, since $\mu = \mu'$, j_0 must fulfill $j_0 < s - 1$. So we are also in case (iv) for M', and $\eta = \eta'$. If we are in case (v) for M, then $\mu = 1$ and E_1 is the only composition factor of M adjacent to the exceptional vertex. So we are either also in case (v) for M', or $M' = E_1$ is simple. If we are in case (v) for M', then $\eta = \eta'$. If M' is simple, then $M' = E_1 = X_3$, and v_3 is nonexceptional, since E_1 is the only composition factor of M that is adjacent to the exceptional vertex. So $\eta' = m - \mu' = m - 1$ according to case (iii) of Definition 2.5, and $\eta = \eta'$. If we are in case (vi) for M, then $\mu = \mu'$ forces j_0 to satisfy $j_0 < s - 2$. Thus we are also in case (vi) for M' and $\eta = \eta'$. Suppose we are in case (vii) for M. Then we are in case (vii) or (vi) for M', or M' is simple. However, case (vi) for M' can occur only if $\gamma > 0$, and we have assumed $\gamma = 0$. If we are in case (vii) for M' then $\eta = \eta'$. If M' is simple, then $M' = E_1 = X_3$. Also v_3 is exceptional, since both $E_1 = X_3$ and $E_2 = X_2$ are adjacent to the exceptional vertex. Note that $E_2 = E_{s-1}$ in case (a) and $E_2 = E_s$ in case (b). So $\eta' = \mu' - 1 = 0$. Since $\mu = \mu' = 1$, it follows that $\eta = \mu - 1 = 0$, so $\eta = \eta'$. Thus it follows in all these cases that $\eta = \eta'$, and so $C_M = (E_s) C_{M'}$.

If $\mu = \mu'$ and W_o occurs, i.e. $\gamma = 1$, then M is as in case (vii) and M' is as in case (vi) of Definition 2.5. So $\eta = \mu - 1$ and $\eta' = \mu' - 2 = \mu - 2$. Thus, according to Theorem 2.6,

$$\begin{aligned} C_M &= (E_s, X_3, X_5, \dots, X_n)(Y_2, Y_4, \dots, Y_{2e})^{\mu-1} \quad \text{and} \\ C_{M'} &= (X_3, X_5, \dots, X_n)(Y_2, Y_4, \dots, Y_{2e})(Y_2, Y_4, \dots, Y_{2e})^{\mu-2} \\ &= (X_3, X_5, \dots, X_n)(Y_2, Y_4, \dots, Y_{2e})^{\mu-1}, \end{aligned}$$

which means $C_M = (E_s)C_{M'}$.

If $\mu \neq \mu'$ and W_o occurs, then M and M' are both as in case (vi) and further $\mu = \mu' + 1$. So $\eta = \mu - 2$ and $\eta' = \mu' - 2 = \mu - 3$, and thus

$$\begin{aligned} C_M &= (E_s, X_3, X_5, \dots, X_n)(Y_2, Y_4, \dots, Y_{2e})^{\mu-2} \quad \text{and} \\ C_{M'} &= (X_3, X_5, \dots, X_n)(Y_2, Y_4, \dots, Y_{2e})(Y_2, Y_4, \dots, Y_{2e})^{\mu-3} \\ &= (X_3, X_5, \dots, X_n)(Y_2, Y_4, \dots, Y_{2e})^{\mu-2}, \end{aligned}$$

which means again $C_M = (E_s)C_{M'}$.

We can now assume that W_o does not occur, i.e. $\gamma = 0$, and that μ and μ' are different. In case (a), the multiplicities μ and μ' can only be different if E_{s-1} or E_s is adjacent to the exceptional vertex. Suppose E_{s-1} is not adjacent to the exceptional vertex. Then $\mu = 1$ and $\mu' = 0$. We are in case (vii) of Definition 2.5 for M, and in case (i) for M' because the only v_i which can be adjacent to the exceptional vertex is v_1. So

$$\begin{aligned} C_M &= (E_s, X_3, X_5, \dots, X_n)(Y_2, Y_4, \dots, Y_{2e})^{\mu-1} \\ &= (E_s, X_3, X_5, \dots, X_n) \quad \text{and} \\ C_{M'} &= (X_3, X_5, \dots, X_n), \end{aligned}$$

which means $C_M = (E_s)C_{M'}$. If E_s is not adjacent to the exceptional vertex, then μ and μ' are different only if $E_{s-2} = E_{s-1}$. In this case, $\mu = \mu' + 1$, and we are in case (vi) for M. We are in case (vii) for M' or M' is simple, since W_o does not occur. If M' is as in (vii) then $\eta' = \mu' - 1$. If M' is simple then $M' = E_1 = X_3$, and v_3 is exceptional, since E_{s-1} is adjacent to the exceptional vertex. So $\eta' = \mu' - 1$ according to case (iii). Thus

$$\begin{aligned} C_M &= (E_s, X_3, X_5, \dots, X_n)(Y_2, Y_4, \dots, Y_{2e})^{\mu-2} \quad \text{and} \\ C_{M'} &= (X_3, X_5, \dots, X_n)(Y_2, Y_4, \dots, Y_{2e})^{\mu'-1} \\ &= (X_3, X_5, \dots, X_n)(Y_2, Y_4, \dots, Y_{2e})^{\mu-2}, \end{aligned}$$

which means $C_M = (E_s)C_{M'}$. If E_{s-1} and E_s are both adjacent to the exceptional vertex, then $\mu = m$, since $\begin{matrix} E_s \\ U \\ E_{s-1} \end{matrix}$ is a cohook, and $\mu' = 0$. We are in case (iv) for M and in case (i) for M' because the only v_i which is adjacent to the exceptional vertex is v_2. So

$$\begin{aligned} C_M &= (E_s, X_3, X_5, \dots, X_n)(Y_2, Y_4, \dots, Y_{2e})^{m-\mu} \\ &= (E_s, X_3, X_5 \dots, X_n) \quad \text{and} \\ C_{M'} &= (X_3, X_5, \dots, X_n), \end{aligned}$$

which means again $C_M = (E_s)C_{M'}$.

In case (b), the multiplicities μ and μ' can only be different if E_s is adjacent to the exceptional vertex and $E_s = E_{s-1}$. Then $\mu = \mu' + 1$, and we are in case (vi) for M. We are in case (vii) for M' or M' is simple, since W_o does not occur. If M' is as in (vii) then

$$\begin{array}{rcl} C_M & = & (E_s, X_3, X_5, \ldots, X_n)(Y_2, Y_4, \ldots, Y_{2e})^{\mu-2} \quad \text{and} \\ C_{M'} & = & (X_3, X_5, \ldots, X_n)(Y_2, Y_4, \ldots, Y_{2e})^{\mu'-1} \\ & = & (X_3, X_5, \ldots, X_n)(Y_2, Y_4, \ldots, Y_{2e})^{\mu-2}, \end{array}$$

which means $C_M = (E_s)C_{M'}$. If M' is simple then $M = \begin{array}{cc} E_1 & \\ & E_1 \end{array}$ and $M' = E_1$. So $\mu = 2$ and $\mu' = 1$. Since one endpoint of E_1 is a nonexceptional leaf vertex, this implies $e = 1$. Thus $W_M = (v_a, E_1, v_z, E_1, v_a, E_1, v_z)$, since $A = (E_1, E_1)$, and $W_{M'} = (v_a, E_1, v_z)$. Further v_a is exceptional according to Definition 2.3. So $\eta' = \mu' - 1 = \mu - 2 = 0$ according to case (iii) of Definition 2.5. Since $\eta = \mu - 2 = 0$, $C_M = (E_1, E_1)$ and $C_{M'} = (E_1)$, so $C_M = (E_s)C_{M'}$.

2. $M' = M_h$. Then either

$$\text{(a) } M = E_1 \text{ is simple}$$

and no endpoint of E_1 is a nonexceptional leaf vertex, or

$$\text{(b) } M = \begin{array}{ccc} & & E_s \\ N & & U \\ & E_{s-1} & \end{array}$$

where U is uniserial and $\begin{array}{c} E_s \\ U \\ E_{s-1} \end{array}$ is not a cohook, or

$$\text{(c) } M = \begin{array}{ccc} & E_{s-1} & \\ N & & V \\ & & E_s \end{array}$$

where V is uniserial and no endpoint of E_s is a nonexceptional leaf vertex.
Case 2. is treated in a similar way as case 1. $M = M'_c$.

This proves Theorem 2.6. □

Theorem 2.6 gives a formula for the location of a nonprojective indecomposable Λ-module in $\Gamma_s(\Lambda)$, using only $T(\Lambda)$. To determine the relative locations of two nonisomorphic Λ-modules M and N in $\Gamma_s(\Lambda)$, we also have to find the distance between two hooks belonging to the same boundary of $\Gamma_s(\Lambda)$.

The following result determines when two hooks belong to the same boundary, and the length of a minimal path between two hooks belonging to the same boundary. If $e > 1$ then a hook H is uniquely determined by $E_1 = \mathrm{soc}(H)$ and $E_2 = \mathrm{top}(H)$. Further E_1 is one of the edges which are the next clockwise edges to E_2 around the endpoints of E_2.

Proposition 2.8. *If $e = 1$ then $m > 1$, since we have assumed that $\Gamma_s(\Lambda)$ contains at least two modules. In this case there is exactly one vertex at each of the two boundaries of $\Gamma_s(\Lambda)$. Suppose now that $e > 1$. Let H be a hook with $E_1 = \mathrm{soc}(H)$ and $E_2 = \mathrm{top}(H)$. Let $W_H = (v_1, X_1, v_2, X_2, \ldots, X_{2e}, v_{2e+1})$ be a complete clockwise walk of multiplicity 1 around $T(\Lambda)$ with $X_1 = E_2$ and $X_2 = E_1$.*

A hook H' with $E'_1 = \operatorname{soc}(H')$ and $E'_2 = \operatorname{top}(H')$ belongs to the same boundary of $\Gamma_s(\Lambda)$ as H if and only if there exists $1 \le j \le e$ with $E'_1 = X_{2j}$ and $E'_2 = X_{2j-1}$. Further the length of a minimal path from H to H' is given by $2(j-1)$. Note that the length of a minimal path from H' to H is then $2e - 2(j-1)$.

Proof. We can assume that $e \ge 2$. We show that the hook K with $\operatorname{soc}(K) = X_4$ and $\operatorname{top}(K) = X_3$ belongs to the same boundary of $\Gamma_s(\Lambda)$ as H, and that the length of a minimal path from H to K is 2. Then it follows by induction that all hooks H' with $\operatorname{soc}(H') = X_{2j}$ and $\operatorname{top}(H') = X_{2j-1}$ for some $1 \le j \le e$ belong to the same boundary as H, and that the length of a minimal path from H to H' is given by $2(j-1)$. Since we get e different hooks this way and since each boundary of $\Gamma_s(\Lambda)$ consists exactly of e hooks, also the other direction of the statement follows.

To show that K belongs to the same boundary as H, we determine the almost split sequence starting in H. Note that v_3 is the endpoint of E_1 which is not an endpoint of E_2. If v_3 is a leaf vertex and v_3 is not exceptional, then $X_3 = E_1$ and X_4 is the next clockwise edge to E_1 around $v_4 = v_2$. So $H = \begin{array}{c} E_2 \\ U \\ E_1 \end{array}$ where U is uniserial or zero, and $\operatorname{soc}(\begin{array}{c} E_2 \\ U \end{array}) = X_4$. Then $K = \begin{array}{c} E_1 \\ E_2 \\ U \end{array}$, and the almost split sequence starting in H has the form (up to isomorphism)

$$0 \to H \to \begin{array}{c} E_2 \\ U \end{array} \oplus P_{E_1} \to K \to 0$$

where P_{E_1} is the projective cover of E_1.

Otherwise, v_3 is not a leaf vertex or v_3 is exceptional of multiplicity $m > 1$. Then H and K look like $H = \begin{array}{c} E_2 \\ U \\ E_1 \end{array}$ (respectively $H = E_1 = E_2$ if v_2 is a nonexceptional leaf vertex), and $K = \begin{array}{c} X_3 \\ V \\ X_4 \end{array}$ (respectively $K = X_3 = X_4$ if v_4 is a nonexceptional leaf vertex), where U and V are uniserial or zero. Then the almost split sequence starting in H is given by (up to isomorphism)

$$0 \to H \to \begin{array}{ccccc} & & & & E_2 \\ & & X_3 & U & \\ & V & & E_1 & \\ X_4 & & & & \end{array} \to K \to 0,$$

where $\begin{array}{c} X_3 \\ V \\ X_4 \end{array}$ has to be replaced by X_3 if K is simple, respectively $\begin{array}{c} E_2 \\ U \\ E_1 \end{array}$ has to be replaced by E_1 if H is simple. Thus K belongs to the same boundary of $\Gamma_s(\Lambda)$ as H, and the length of a minimal path from H to K is 2. This proves Proposition 2.8. □

Corollary 2.9. *With the aid of Theorem 2.6 and Proposition 2.8, we can determine the relative locations of any two nonprojective indecomposable Λ-modules M and N in $\Gamma_s(\Lambda)$, by looking at certain walks around $T(\Lambda)$.*

The following result gives an explicit formula for the length of a shortest path from M to N in $\Gamma_s(\Lambda)$, provided we know the distances of M, respectively N, to the same boundary as given by Theorem 2.6, and the distance between two hooks belonging to the same boundary as given by Proposition 2.8.

Proposition 2.10. *Let M and N be indecomposable Λ-modules in $\Gamma_s(\Lambda)$. Suppose H_M and H_N are two hooks belonging to the same boundary of $\Gamma_s(\Lambda)$ such that there exist directed paths from M to H_M of length d_M, respectively from N to H_N of length d_N. Let further d_{MN} be the length of a shortest path from H_M to H_N. Then the length d of a shortest path from M to N is given by*

$$d = 2e \cdot l + d_{MN} + d_M - d_N$$

where l is an integer defined as follows. If $d_M \geq d_N$, or $d_M < d_N$ and $2(d_N - d_M) \leq d_{MN}$, then $l = 0$. If $d_M < d_N$ and $2(d_N - d_M) > d_{MN}$, then $l \geq 1$ is the integer such that $2e \cdot (l-1) \leq 2(d_N - d_M) - d_{MN} < 2e \cdot l$.

Proof. We define the boundary of $\Gamma_s(\Lambda)$ containing H_M and H_N to be the upper boundary, and the other boundary to be the lower boundary of $\Gamma_s(\Lambda)$. Further, each directed path heading toward the upper boundary (respectively lower boundary) is called an up-path (respectively down-path). If $d_M \geq d_N$ then M is below N. Suppose M' is the module at the end of the up-path of length $d_M - d_N$ starting at M. Then $d_{M'} = d_N$ and the shortest paths from M' to N have length d_{MN}. So $d = (d_M - d_N) + d_{MN} = d_{MN} + d_M - d_N$. If $d_M < d_N$ then M is above N. Let M' be the module at the end of the down-path of length $d_N - d_M$ starting at M, which means $d_{M'} = d_N$. Suppose first that $2(d_N - d_M) \leq d_{MN}$. Then the shortest paths from M' to N are by $2(d_N - d_M)$ shorter than the shortest paths from H_M to H_N. So $d = (d_N - d_M) + (d_{MN} - 2(d_N - d_M)) = d_{MN} + d_M - d_N$. Let now $2(d_N - d_M) > d_{MN}$. Then there is a path from N to M' of length $2(d_N - d_M) - d_{MN}$. Since $d_{M'} = d_N$, the minimal length of a path from N to M' has to be less than $2e$. So, with l as in the statement of Proposition 2.10, $2(d_N - d_M) - d_{MN} - 2e \cdot (l-1)$ is the length of the shortest paths from N to M'. Thus the length of a shortest path from M' to N is $2e - [2(d_N - d_M) - d_{MN} - 2e \cdot (l-1)] = 2e \cdot l + d_{MN} - 2(d_N - d_M)$. Therefore it follows that $d = (d_N - d_M) + (2e \cdot l + d_{MN} - 2(d_N - d_M)) = 2e \cdot l + d_{MN} + d_M - d_N$. This proves Proposition 2.10.

□

3. Stable endomorphism rings

In this section we give an explicit description of the stable endomorphism ring of a nonprojective indecomposable Λ-module M, using the description of M via the top-socle path and the multiplicity of M as given in Definition 2.1. Note that this section is independent of the results proved in Section 2.

We need the following lemma.

Lemma 3.1. *Let I_M be a set of representatives for the isomorphism classes of indecomposable Λ-modules which are both factor modules and submodules of M. If $Q \in I_M$ and $Q \not\cong M$ then Q is uniserial, and each composition factor of Q occurs at least twice in M (but not necessarily twice in Q).*

Proof. This is clear from the description of the indecomposable factor modules and submodules of M given in [15, Prop. 2.1]. □

For each $Q \in I_M$ we now choose a surjection $\kappa_Q : M \to Q$ and an injection $\iota_Q : Q \to M$. We will call $\alpha_Q = \kappa_Q \iota_Q$ the chosen endomorphism of M factoring through Q. In the following we will choose α_M to be id_M. By [8, 13] and [15, Prop. 2.1], every endomorphism α of M is given as a k-linear combination of chosen endomorphisms α_Q as Q runs over I_M.

Gabriel and Riedtmann proved in [9] that $\underline{\mathrm{End}}_\Lambda(M)$ has a nilpotent generator as a k-algebra. In the following result we show how to determine an explicit nilpotent generator, using only the top-socle path and the multiplicity of M. The nilpotency of this generator is computed in Proposition 3.5.

Theorem 3.2. *Let M be a nonprojective indecomposable Λ-module with top-socle path $E_1, \dots, E_s$ and multiplicity μ, and let $\pi : \mathrm{End}_\Lambda(M) \to \underline{\mathrm{End}}_\Lambda(M)$ be the natural surjection. Suppose $\rho \in \underline{\mathrm{End}}_\Lambda(M)$ is given as follows.*

(i) *If M has only composition factors with multiplicity 1, then $\rho = 0$.*

(ii) *If M has composition factors of multiplicity greater than 1, then $\mu \geq 2$. Let S_1 (respectively S_j) be the unique composition factor of $\mathrm{top}(M)$ (respectively $\mathrm{soc}(M)$) corresponding to an edge in $T(\Lambda)$ which is adjacent to the exceptional vertex. Denote all the edges which are adjacent to the exceptional vertex by $S_1, \dots, S_j, \dots, S_n$ in counterclockwise order. Let the descending radical series of the uniserial part of M consisting of these modules be given as*
$$V = (S_1, \dots, S_n, S_1, \dots, S_n, S_1, \dots, S_1, \dots, S_j)$$
where S_1 occurs μ times as composition factor of V. Then
$$Q_0 = V/\mathrm{rad}(\Lambda)^{(\mu-2)n+j} V = (S_1, \dots, S_n, S_1, \dots, S_1, \dots, S_j)$$
can be taken to be in I_M, where S_1 occurs $\mu-1$ times in Q_0. Define $\rho = \pi(\rho_0)$, where $\rho_0 = \alpha_{Q_0}$ is the chosen endomorphism of M factoring through Q_0. We will call ρ_0 a "shift" of M.

Then $\rho \in \mathrm{rad}(\underline{\mathrm{End}}_\Lambda(M))$, and there exists a positive integer $r \leq \mu$ with $\rho^{r-1} \neq 0$ and $\rho^r = 0$ in $\underline{\mathrm{End}}_\Lambda(M)$ such that
$$\underline{\mathrm{End}}_\Lambda(M) = k \cdot \pi(\mathrm{id}_M) \oplus k \cdot \rho \oplus k \cdot \rho^2 \oplus \cdots \oplus k \cdot \rho^{r-1}$$
as k-vector space.

Remark 3.3. Suppose $\tilde{\rho}_0$ is an endomorphism of M factoring through Q_0. From the form of V and Q_0 one sees that
$$\tilde{\rho}_0 = \sum_{l=1}^{\mu-1} a_l \cdot \rho_0^l$$
for some $a_l \in k$. The $\tilde{\rho}_0$ which can be used as chosen endomorphisms of M factoring through Q_0 are those for which $a_1 \neq 0$.

Proof. If M has only composition factors with multiplicity 1, then by Lemma 3.1 we can choose $I_M = \{M\}$ and α_M to be the identity map id_M. Hence $\underline{\mathrm{End}}_\Lambda(M) = k \cdot \pi(\mathrm{id}_M)$.

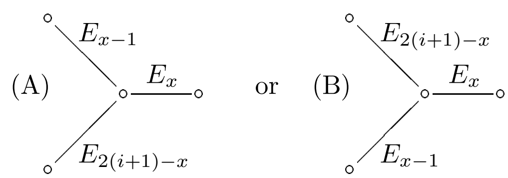
Suppose now that M has composition factors of multiplicity greater than 1. Let $E_1, \dots, E_s$, μ, $S_1, \dots, S_n$, Q_0, ρ_0 and ρ be as in the statement of Theorem 3.2(ii).

We want to show that ρ generates $\mathrm{rad}(\underline{\mathrm{End}}_\Lambda(M))$. Suppose first that $\mathrm{soc}(M) \cap \mathrm{top}(M) = \emptyset$. Then all composition factors of M that are not isomorphic to $S_1, \dots, S_n$ have multiplicity 1. So by Lemma 3.1, the only $Q \ncong M$ in I_M are factor modules of Q_0 having S_j as socle. The associated α_Q factor through Q_0, so it follows from Remark 3.3 that ρ generates $\mathrm{rad}(\underline{\mathrm{End}}_\Lambda(M))$.

We now suppose $\mathrm{soc}(M) \cap \mathrm{top}(M) \neq \emptyset$. Then S_j must be equal to S_1, and we might have to consider additional Q. To describe this special phenomenon, we consider the top-socle path $E_1, \ldots, E_s$. We are in the situation that there exists an i with $E_i = E_{i+1} = S_1$. Then there exists a smallest $1 \leq x \leq i$ with $E_{i-u} = E_{i+1+u}$ for $0 \leq u \leq i - x$, and $E_{x-1} \neq E_{2(i+1)-x}$. Thus $E_1, \ldots, E_s$ looks like

$$E_1, \ldots, E_{x-1}, E_x, \ldots, E_{i-1}, E_i, E_i, E_{i-1}, \ldots, E_x, E_{2(i+1)-x}, \ldots, E_s.$$

Note that the composition factors of M lying between $E_1, \ldots, E_{x-1}$ or between $E_{2(i+1)-x}, \ldots, E_s$, excluding E_{x-1} and $E_{2(i+1)-x}$, have multiplicity 1. By Lemma 3.1, the additional Q are uniserial, and have all composition factors occurring at least twice in M. Further, $\mathrm{top}(Q)$ and $\mathrm{soc}(Q)$ are elements of $\{E_{x-1}, \ldots, E_{2(i+1)-x}\}$. By considering which composition factors of M have multiplicity at least 2, we conclude that the $Q \in I_M$ which are not isomorphic to M or factor modules of Q_0 are the simple modules $E_x, \ldots, E_{i-1}$, and possibly a module X involving E_{x-1} and $E_{2(i+1)-x}$. To see whether X exists or not, we have to look at the following cases. Either E_{x-1} belongs to $\mathrm{top}(M)$, or E_{x-1} belongs to $\mathrm{soc}(M)$. The subtree of $T(\Lambda)$ containing the edges E_x, E_{x-1} and $E_{2(i+1)-x}$ can be either of the following two possibilities:



If E_{x-1} belongs to $\mathrm{top}(M)$, X exists only if the subtree is as in (A). Then X is the uniserial module obtained by walking counterclockwise from E_{x-1} to $E_{2(i+1)-x}$. In case that E_{x-1} belongs to $\mathrm{soc}(M)$, then the subtree must be as in (B), and X is the uniserial module obtained by walking counterclockwise from $E_{2(i+1)-x}$ to E_{x-1}.

We now show that we can select the additional chosen endomorphisms $\alpha_{E_{i-1}}, \ldots, \alpha_{E_x}, \alpha_X$ so that each can be written as a sum

$$\psi_U + \sum_{l=1}^{\mu-1} b_l \rho_0^l$$

where ψ_U is an endomorphism of M which factors through a projective module and $b_l \in k$. By Remark 3.3 and the paragraph just prior to Theorem 3.2, this will suffice to prove Theorem 3.2.

Let $\psi_{E_{i-1}}$ be an endomorphism of M factoring through the projective cover P_{E_i} of E_i in the following way:

$$M \to \begin{array}{ccc} E_{i-1} & & E_i \\ & \ddots \quad \cdot^{\cdot^{\cdot}} & \\ & E_i & \end{array} \to P_{E_i} \to \begin{array}{ccc} & E_i & \\ \cdot^{\cdot^{\cdot}} & & \ddots \\ E_{i-1} & & E_i \end{array} \to M$$

We assume the sequence of maps defining $\psi_{E_{i-1}}$ is surjective, injective, surjective and then injective. The image of $\psi_{E_{i-1}}$ is isomorphic to $E_{i-1} \oplus V_0$ where V_0 is a factor module of Q_0 such that $\mathrm{soc}(V_0) = \mathrm{top}(V_0) = S_1$. Thus we can choose

$E_{i-1} \in I_M$ and $\alpha_{E_{i-1}}$ in such a way that $\psi_{E_{i-1}} - \alpha_{E_{i-1}}$ factors through V_0. Remark 3.3 now shows

$$\alpha_{E_{i-1}} = \psi_{E_{i-1}} - \sum_{l=1}^{\mu-1} c_l \rho_0^l \quad \text{for some } c_l \in k$$

as required.

We will define α_{E_u} for $x \leq u \leq i-2$ by descending induction on u. Let $\alpha_{E_i} = \rho_0^{\mu-1}$, and suppose that $\alpha_{E_{u+2}}$ has already been defined. Since E_{u+2} is simple, all endomorphisms of M factoring through E_{u+2} are scalar multiples of $\alpha_{E_{u+2}}$. Let ψ_{E_u} be given as

$$M \to \begin{array}{ccc} E_u & & E_{u+2} \\ & \ddots \quad \iddots & \\ & E_{u+1} & \end{array} \to P_{E_{u+1}} \to \begin{array}{ccc} & E_{u+1} & \\ & \iddots \quad \ddots & \\ E_u & & E_{u+2} \end{array} \to M$$

Then the image of ψ_{E_u} is $E_u \oplus E_{u+2}$. Therefore after multiplying ψ_{E_u} by a suitable nonzero scalar, we will have

$$\alpha_{E_u} = \psi_{E_u} - \alpha_{E_{u+2}}$$

for some α_{E_u}. This suffices by induction to show that all α_{E_u} with $x \leq u \leq i-2$ may be chosen to have the required form.

Finally we must define a suitable endomorphism α_X corresponding to X. If E_{x-1} belongs to $\operatorname{top}(M)$, then we can construct an endomorphism ψ_X which factors through the projective module P_{E_x}:

$$M \to \begin{array}{ccc} E_{x-1} & & E_{x+1} \\ & \ddots \quad \iddots & \\ & E_x & \end{array} \to P_{E_x} \to \begin{array}{ccc} & E_x & \\ & \iddots \quad \ddots & \\ E_{2(i+1)-x} & & E_{x+1} \end{array} \to M$$

If $x < i$, then the image of ψ_X is $X \oplus E_{x+1}$, and $\alpha_{E_{x+1}}$ has already been defined (and is unique up to multiplication by a nonzero scalar). In this case, replacing ψ_X by a suitable nonzero multiple of itself, we may define

$$\alpha_X = \psi_X - \alpha_{E_{x+1}}$$

which is sufficient to complete the proof. Otherwise the image of ψ_X is isomorphic to $X \oplus V_1$ where V_1 is a quotient module of Q_0 with $\operatorname{soc}(V_1) = \operatorname{top}(V_1) = S_1$. In this case we can define α_X so $\psi_X - \alpha_X$ factors through V_1. Then

$$\alpha_X = \psi_X - \sum_{l=1}^{\mu-1} c_l \rho_0^l \quad \text{for some } c_l \in k$$

which suffices to complete the proof. If E_{x-1} belongs to $\operatorname{soc}(M)$, then ψ_X is constructed as follows:

$$M \to \begin{array}{ccc} E_{2(i+1)-x} & & E_{x+1} \\ & \ddots \quad \iddots & \\ & E_x & \end{array} \to P_{E_x} \to \begin{array}{ccc} & E_x & \\ & \iddots \quad \ddots & \\ E_{x-1} & & E_{x+1} \end{array} \to M$$

The analysis of the cases $x < i$ and $x = i$ is now similar using this ψ_X. $\square$

Remark 3.4. The proof of Theorem 3.2, using [8, 13], shows also how to compute the endomorphism ring of M. In case (i)

$$\mathrm{End}_\Lambda(M) = k \cdot \mathrm{id}_M.$$

In case (ii), if $\mathrm{soc}(M) \cap \mathrm{top}(M) = \emptyset$ then

$$\mathrm{End}_\Lambda(M) = k \cdot \mathrm{id}_M \oplus k \cdot \rho_0 \oplus k \cdot \rho_0^2 \oplus \cdots \oplus k \cdot \rho_0^{\mu-1}.$$

If $\mathrm{soc}(M) \cap \mathrm{top}(M) \neq \emptyset$ then

$$\mathrm{End}_\Lambda(M) = k \cdot \mathrm{id}_M \oplus k \cdot \rho_0 \oplus k \cdot \rho_0^2 \oplus \cdots \oplus k \cdot \rho_0^{\mu-1} \oplus k \cdot \alpha_{E_{i-1}} \oplus \cdots \oplus k \cdot \alpha_{E_x} (\oplus k \cdot \alpha_X)$$

where α_X is only included if X exists.

We now want to give an explicit formula for the nilpotency r of ρ as defined in Theorem 3.2(ii), which follows directly from the top-socle path and the multiplicity of M.

Proposition 3.5. *Let M be given as in Theorem 3.2(ii) such that S_1 occurs $\mu \geq 2$ times in*

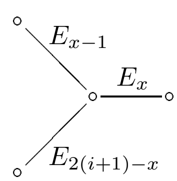
$$V = (S_1, \ldots, S_n, S_1, \ldots, S_n, S_1, \ldots, S_1, \ldots, S_j)$$

and S_1 lies in $\mathrm{top}(M)$ and S_j lies in $\mathrm{soc}(M)$. Let E_i be the edge in the top-socle path with $E_i = S_1$. By taking the mirror image of M if necessary, we assume that $E_{i+1} = S_j$. Let r be defined as follows.

(i) *If $S_1 \neq S_j$, then $r = \min(\mu, m + 1 - \mu)$.*

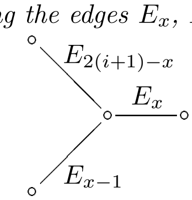
Suppose now that $S_1 = S_j$. Then $E_i = E_{i+1} = S_1$, and there exists a smallest $1 \leq x \leq i$ with $E_{i-u} = E_{i+1+u}$ for $0 \leq u \leq i - x$, and $E_{x-1} \neq E_{2(i+1)-x}$ (if they exist).

(ii) *If ($x = 1$ or $2i + 1 - x = s$) and $E_x \in \mathrm{top}(M)$, or if $x > \max(1, 2i + 1 - s)$ and the subtree of $T(\Lambda)$ containing the edges E_x, E_{x-1} and $E_{2(i+1)-x}$ looks like*



then $r = \min(\mu, m + 1 - \mu)$.

(iii) *If ($x = 1$ or $2i+1-x = s$) and $E_x \in \mathrm{soc}(M)$, or if $x > \max(1, 2i+1-s)$ and the subtree of $T(\Lambda)$ containing the edges E_x, E_{x-1} and $E_{2(i+1)-x}$ looks like*



then $r = \min(\mu - 1, m + 2 - \mu)$.

Proof. The strategy of the proof is to analyze for each simple Λ-module T whether there is an endomorphism of M factoring through the projective cover P_T, and to write down all such endomorphisms in terms of the basis $\mathcal{B}$ of $\mathrm{End}_\Lambda(M)$ found in Remark 3.4. In the proof of Theorem 3.2 we analyzed how to write each element of $\mathcal{B}$ as a sum $\psi_U + \beta_U$, where ψ_U factors through a projective module and β_U is an explicit linear combination of powers of ρ_0. We can in this way analyze the minimal exponent t for which some linear combination $\sum_{l=t}^{\mu-1} a_l \rho_0^l$ with $a_t \neq 0$ factors through a projective module. This t is equal to the nilpotency r.

We first consider case (i). Then $\mathrm{soc}(M) \cap \mathrm{top}(M) = \emptyset$ and all endomorphisms of M are linear combinations of powers of ρ_0. It follows that the smallest power of ρ_0 which factors through a projective module factors through some P_T with $T \in \{S_1, \ldots, S_n\}$ or T adjacent to one of the S_l. By considering these possibilities, and using the fact that $\mathrm{Aut}_\Lambda(M)$ acts on $\underline{\mathrm{End}}_\Lambda(M)$, we see that $T = S_j$ provides $r = \min(\mu, m + 1 - \mu)$.

In cases (ii) and (iii), one finds r is realized by the following construction.

In case (ii), let C be the factor module of M with top-socle path S_1, S_1, E_{i+2} (where we include E_{i+2} only if $i < s - 1$), and let D be the submodule of M with top-socle path E_{i-1}, S_1, S_1 (where we include E_{i-1} only if $i > 1$). Then there exists an endomorphism

$$\psi : M \to C \to P_{S_1} \to D \to M$$

which has as image $E_{i-1} \oplus V'$ where $V' = (S_1, \ldots, S_n, S_1, \ldots, S_1)$ with S_1 occurring $(2\mu - m - 1)$ times if $2\mu > m + 1$, and $V' = 0$ otherwise. From the proof of Theorem 3.2 it follows that the chosen endomorphism $\alpha_{E_{i-1}}$ factoring through E_{i-1} factors through a projective module. In case $2\mu > m + 1$ this means that $\psi - \alpha_{E_{i-1}}$ is of the form $\sum_{l=m+1-\mu}^{\mu-1} c_l \rho_0^l$ for suitable $c_l \in k$ with $c_{m+1-\mu} \neq 0$. It follows that $r = m + 1 - \mu$ if $2\mu > m + 1$, and $r = \mu$ otherwise.

In case (iii), the chosen endomorphism factoring through E_{i-1} does not factor through a projective module. So in that case we let D' be the submodule of M with top-socle path S_1, S_1 such that S_1 occurs $\mu - 1$ times. Then the image of

$$\psi' : M \to C \to P_{S_1} \to D' \to M$$

is nonzero exactly if $2\mu - 1 > m + 1$. In that case $\psi' = \sum_{l=m+2-\mu}^{\mu-1} c_l \rho_0^l$ for suitable $c_l \in k$ with $c_{m+2-\mu} \neq 0$. Further, it follows from the proof of Theorem 3.2 that the chosen endomorphism α_{S_1} factoring through S_1 factors through a projective module. Since $\rho_0^{\mu-1}$ is a nonzero multiple of α_{S_1}, this means that $\rho_0^{\mu-1}$ factors through a projective module. So we obtain $r = m + 2 - \mu$ if $2\mu - 1 > m + 1$, and $r = \mu - 1$ otherwise.

This completes the proof of Proposition 3.5 □

4. Hom-paths and Ext groups

In this section we discuss applications of the two previous section. The main result is Theorem 4.5. This gives a description of how one can compute stable Hom groups and Ext groups using Theorems 3.2 and 2.6, and Propositions 2.8 and 2.10.

To prove Theorem 4.5 we will use some ideas and methods of Gabriel and Riedtmann [9].

Definition 4.1. Let $l \geq 0$ and let $(M = X_0, X_1, \ldots, X_l = N)$ be a path of successively nonisomorphic Λ-modules in $\Gamma_s(\Lambda)$. If $l \geq 1$ and $f_i : X_{i-1} \to X_i$, $1 \leq i \leq l$, is an arbitrary irreducible morphism, then $f = f_1 \cdots f_l$ is called a *hom-path* of length $l = l(f)$ from M to N, corresponding to $(M = X_0, \ldots, X_l = N)$. In case that $l = 0$, a hom-path f of length 0 from M to M, corresponding to the path (M), is given by an automorphism of M.

The next two theorems follow directly from the methods of Gabriel and Riedtmann in [9], though they do not appear explicitly in this form in [9]. For this reason we will state them without proof.

Theorem 4.2. (Gabriel-Riedtmann) *Let M and N be two nonprojective indecomposable Λ-modules, and let $f \in \underline{\mathrm{Hom}}_\Lambda(M,N)$ be a hom-path of minimal length. Then*

$$\underline{\mathrm{Hom}}_\Lambda(M,N) = \underline{\mathrm{End}}_\Lambda(M)\cdot f = f\cdot \underline{\mathrm{End}}_\Lambda(N).$$

Suppose ρ is an ideal generator of $\mathrm{rad}(\underline{\mathrm{End}}_\Lambda(M))$ and σ is an ideal generator of $\mathrm{rad}(\underline{\mathrm{End}}_\Lambda(N))$ as given in Theorem 3.2. Then there exists a positive integer $j(f)$ such that

$$\begin{aligned}\underline{\mathrm{Hom}}_\Lambda(M,N) &= k\cdot f \oplus k\cdot \rho f \oplus \cdots \oplus k\cdot \rho^{j(f)-1} f\\ &= k\cdot f \oplus k\cdot f\sigma \oplus \cdots \oplus k\cdot f\sigma^{j(f)-1}\end{aligned}$$

as k-vector space.

Theorem 4.3. (Gabriel-Riedtmann) *Let M and N be nonprojective indecomposable Λ-modules. Let d_1 (respectively d_2) be the length of a shortest hom-path from $\Omega^{-1}(N)$ to M (respectively from M to N). Then $me-1 = d_1+d_2+2e\cdot s$ for some integer s, and $\dim_k \underline{\mathrm{Hom}}_\Lambda(M,N) = \max(0, s+1)$.*

For arbitrary nonprojective indecomposable Λ-modules M and N, the Ext groups $\mathrm{Ext}^i_\Lambda(M,N)$ have the form

$$\mathrm{Ext}^i_\Lambda(M,N) \cong \begin{cases} \mathrm{Hom}_\Lambda(M,N) & ; \quad i=0\\ \underline{\mathrm{Hom}}_\Lambda(\Omega^i(M),N) & ; \quad i>0\end{cases}$$

Note that $\underline{\mathrm{Hom}}_\Lambda$ can be replaced by Hom_Λ if M or N are simple.

So one has as corollary of Theorem 4.3 the following result.

Corollary 4.4. *Suppose M and N are nonprojective indecomposable Λ-modules and $i > 0$. Let d_1 (respectively d_2) be the length of a shortest hom-path from $\Omega^{-1}(N)$ to $\Omega^i(M)$ (respectively from $\Omega^i(M)$ to N). Then $me-1 = d_1+d_2+2e\cdot s$ for some integer s, and $\dim_k \mathrm{Ext}^i_\Lambda(M,N) = \max(0,s+1)$.*

We now combine these results with the work of the previous sections. The multi-pushout description of $\Omega^i(M)$ follows from the one of M for all integers i. Thus we get the following theorem:

Theorem 4.5. *The dimensions $\dim_k \underline{\mathrm{Hom}}_\Lambda(M,N)$ and $\dim_k \mathrm{Ext}^i_\Lambda(M,N)$ for $i>0$ can be found from the multi-pushout descriptions of M and N in the following way. Theorem 2.6 and Propositions 2.8 and 2.10 provide a method to compute the length d_1 (respectively d_2) of a shortest hom-path from $\Omega^{-1}(N)$ to $\Omega^i(M)$ (respectively from $\Omega^i(M)$ to N) for $i \geq 0$, using only calculations in $T(\Lambda)$. Theorem 4.3 and Corollary 4.4 can then be used to compute the above dimensions using only d_1 and d_2.*

Remark 4.6. Using only the location of M in $\Gamma_s(\Lambda)$, the location of $\Omega(M)$ in $\Gamma_s(\Lambda)$ can be described explicitly as follows.

Let M be a nonprojective indecomposable Λ-module. If M lies at one of the boundaries of $\Gamma_s(\Lambda)$, then $\Omega(M)$ is the module at the end of the maximal directed path starting at M. If M is not at one of the boundaries, then $\Omega(M)$ is the module at the end of a path of length $me-1$ given by the composition of a maximal directed path starting at M with a directed path.

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