

Duality and Hermitian Galois module structure

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Abstract

Suppose \mathcal{X} is a regular projective scheme which is flat and equidimensional over \mathbf{Z} of relative dimension d . Let G be a finite group acting tamely on \mathcal{X} . Define $HCl(\mathbf{Z}G)$ to be the Hermitian class group of $\mathbf{Z}G$. Using the duality pairings on the de Rham cohomology groups $H^*(X, \Omega_{X/\mathbf{Q}}^\bullet)$ of the generic fiber X of \mathcal{X} , we define a canonical invariant $\chi_H(\mathcal{X}, G)$ in $HCl(\mathbf{Z}G)$ whose image in $Cl(\mathbf{Z}G)$ is $(-1)^d$ times the de Rham invariant $\chi(\mathcal{X}, G)$ considered in [4]. When \mathcal{X} is a curve over \mathbf{Z} we determine the image of $\chi_H(\mathcal{X}, G)$ in the adelic Hermitian classgroup $Ad\ HCl(\mathbf{Z}G)$ by means of ϵ -constants. We also show that in this case, the image in $Ad\ HCl(\mathbf{Z}G)$ of a closely related Hermitian Euler characteristic $\chi_H(\mathcal{X}, G)(0)$ both determines and is determined by the ϵ_0 -constants of the symplectic representations of G .

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1 Introduction.

Suppose \mathcal{X} is a regular projective scheme over \mathbf{Z} and that G a finite group acting on \mathcal{X} . An $\mathcal{O}_{\mathcal{X}}-G$ module is a coherent $\mathcal{O}_{\mathcal{X}}$ -module having an action of G compatible with the action of G on $\mathcal{O}_{\mathcal{X}}$. Coherent Galois module structure theory has to do with the G -module structure of the cohomology groups of bounded complexes L^{\bullet} of $\mathcal{O}_{\mathcal{X}}-G$ modules. The main goal of this paper is to develop a theory which also measures in a suitable sense the isomorphisms and pairings provided by the coherent duality theorem of [19]. One application is to the study of equivariant de Rham discriminants and their connection to epsilon factors. This can be viewed as an equivariant generalization of work of Bloch in [1].

A natural approach to studying the cohomology of L^{\bullet} is to define an Euler characteristic in a suitable Grothendieck group of finitely generated G -modules. If the action of G on \mathcal{X} is tame, in the sense of [5], then one can define such an Euler characteristic $\chi^P(L^{\bullet})$ in the Grothendieck group $K_0(\mathbf{Z}G)$ of all finitely generated projective $\mathbf{Z}G$ -modules. A survey of some results about $\chi^P(L^{\bullet})$ is given in [11]. Many of these results were motivated by the study of the stable isomorphism class of rings of algebraic integers as Galois modules (c.f. [12]). The fundamental result about tame rings of integers is Taylor's theorem [26], which links the class in $K_0(\mathbf{Z}G)$ of the ring of integers of a finite tame Galois extension of number fields to the Artin root numbers of symplectic representations of

the Galois group G of the extension. A generalization of this Theorem to the de Rham cohomology of all \mathcal{X} as above satisfying certain natural conditions was proved in [4, 6].

Various Euler characteristics in more refined Grothendieck groups than $K_0(\mathbf{Z}G)$ have been studied over the years. The Hermitian classgroup $HCl(\mathbf{Z}G)$ of $\mathbf{Z}G$, and a quotient $Ad HCl(\mathbf{Z}G)$ of $HCl(\mathbf{Z}G)$ called the adelic Hermitian class group, were defined by Fröhlich in order to study the Galois structure of rings of integers together with the natural trace form. This led eventually to the results of Cassou-Noguès and Taylor in [2]. They proved that the adelic Hermitian classes of tame rings of integers together with their trace forms are both determined by and determine local and global root number of symplectic representations of Galois groups.

There are (at least) two very different ways of generalizing Hermitian theory of rings of integers to coherent sheaves on schemes. One method is to introduce G -invariant Hermitian metrics on such sheaves, as in Arakelov theory, and to study G -equivariant metrized Euler characteristics. This approach is taken in [7], in which an Arakelov theoretic generalization of Taylor's theorem for \mathcal{X} of arbitrary dimension is proved. The approach we will take in this paper is different, and is motivated by the fact that the trace form on rings of integers arises from the coherent duality theorem for schemes of dimension 1.

The first step in pursuing Hermitian Galois structure via coherent duality is to associate an Euler characteristic in Fröhlich's Hermitian class group $HCl(\mathbf{Z}G)$ to a perfect complex P^\bullet of $\mathbf{Z}G$ -modules for which one has a suitable family \langle, \rangle of duality pairings in cohomology. We define such an Euler characteristic $d(P^\bullet, \langle, \rangle)$ in §2. Following Fröhlich, we call $d(P^\bullet, \langle, \rangle)$ a Hermitian discriminant. Our main result concerning this discriminant, Theorem 2.8, is that it depends only on the isomorphism class of P^\bullet in the derived category together with the family of pairings \langle, \rangle . The proof uses a method of computing Pfaffians which is postponed until Appendix C. In Appendix D we complete the proof of Theorem 2.8 by showing the homotopy invariance of discriminants of complexes with pairings.

In §3 we consider arithmetic applications to schemes \mathcal{X} as above having a tame action by a finite group G . We define a canonical Hermitian de Rham discriminant $\chi_H(\mathcal{X}, G)$ in $HCl(\mathbf{Z}G)$ whose image in $Cl(\mathbf{Z}G)$ is the de Rham invariant considered in [4, 6]. A construction due to Dold and Puppe of exterior power functors on perfect complexes of $\mathcal{O}_{\mathcal{X}}$ -modules is needed to create suitable canonical complexes on whose cohomology groups one has the appropriate duality pairings.

The rest of this paper is devoted to relating $\chi_H(\mathcal{X}, G)$, and a variant $\chi_H(\mathcal{X}, G)(0)$ of this discriminant, to ϵ -factors of representations when $\dim(\mathcal{X}) = 2$. Our main results provide Hermitian generalizations of the work on surfaces in [4].

In Theorem 6.4 we show that the image of $\chi_H(\mathcal{X}, G)(0)$ in the adelic Hermitian class group $Ad HCl(\mathbf{Z}G)$ both determines and is determined by the ϵ_0 constants associated by Deligne to the symplectic representations of G . This is a counterpart for surfaces of the main global result of Cassou-Noguès and Taylor in [2].

It is remarkable that the ϵ -factor class $\epsilon_0^s(\mathcal{Y})^{-1}$ arising in Theorem 6.4 is the same class appearing in the main result proved via the Arakelov approach to metrizing Euler characteristics in [7]. We see no direct reason for this to be true, e.g. because the metrics on cohomology in the Arakelov approach are positive definite while the pairings on cohomology we use in this paper will not in general be so.

Our second result concerning surfaces, Theorem 6.7, relates the image in $Ad\ HCl(\mathbf{Z}G)$ of the discriminant $\chi_H(\mathcal{X}, G)$ to ϵ -factors. We show that this image is the sum of a global symplectic root number term and a term associated to root numbers over the branch locus of $\pi : \mathcal{X} \rightarrow \mathcal{X}/G$. This mirrors the decompositions of the de Rham Euler characteristics proved in [3, 4, 6]. One interesting feature of this result is that while $\chi_H(\mathcal{X}, G)(0)$ defines a rational class, in the sense of [2] and §5, we do not expect this to be true in general of $\chi_H(\mathcal{X}, G)$.

We end this introduction with a sketch of the proofs of Theorems 6.4 and 6.7 and with a summary of the structure of this paper.

The first step towards proving the main theorems is to reduce to considering characters of degree 0. One shows this reduction is equivalent to proving the theorems when G is the trivial group. In this case, the theorems follow from work of Saito [22] on ϵ_0 factors and of Bloch [1] on the relation between conductors and de Rham discriminants. To treat characters of degree 0, the strategy (as in [4]) is to reduce to the case of rings of integers by a judicious choice of effective horizontal divisors \mathcal{D} and \mathcal{J} on $\mathcal{Y} = \mathcal{X}/G$. The algebraic problem is to show that

$$\tilde{\chi}_H(\mathcal{X}, G)(0) = \tilde{d}(H^0(\mathcal{O}_{\mathcal{D}'}, tr_{\mathcal{D}'}) \cdot \tilde{d}(H^0(\mathcal{O}_{\mathcal{J}'}, tr_{\mathcal{J}'})^{-2} \quad (1.1)$$

where \tilde{c} is the restriction to characters of degree 0 of a class c , $\mathcal{D}' = \pi^{-1}(\mathcal{D})$, $\mathcal{J}' = \pi^{-1}(\mathcal{J})$ and $tr_{\mathcal{D}'}$ and $tr_{\mathcal{J}'}$ are the trace forms on the generic fibers D' and J' of \mathcal{D}' and \mathcal{J}' , respectively. This requires establishing various exact sequences in cohomology and comparing via these sequences the pairings involved in the definition of the discriminants appearing in (1.1). The step of comparing pairings is the main additional complication beyond the work in [4, 6], and it is the main difficulty now in treating schemes of arbitrary dimension.

In §7 we prove the expected result about how discriminants behave when one has a triangle of complexes of $\mathcal{O}_{\mathcal{X}}$ - G -modules in which one term is supported on a finite number of finite fibers. The reduction to characters of degree 0 mentioned above is carried out in §8 and §9. We collect in Appendices A and B the facts about discriminants needed to treat virtual characters of degree 0 in the case of surfaces. In §10 we show that one can make certain base changes to prove Theorem 6.4 which are needed to construct the divisors \mathcal{D} and \mathcal{J} in §11. In §11 we make the computations of trace pairings necessary to establish (1.1). The proof of Theorem 6.4 is then completed in §12 using the main result of Cassou-Noguès and Taylor in [2] to relate the right hand side of (1.1) to root numbers associated to \mathcal{D}' and \mathcal{J}' , and the comparison of these root numbers to those going into the definition of $\tilde{\epsilon}^s(\mathcal{Y})^{-1}$ which was made in [4, §9]. We prove Theorem 6.7 using Theorem 6.4, §7 and the fibral calculations in [4, §8].

2 Symmetric morphisms of modules and complexes

Let G be a finite group, and suppose M is a finitely generated left $\mathbf{Q}G$ -module. Define $\tilde{M} = \text{Hom}_{\mathbf{Q}}(M, \mathbf{Q})$. We may identify M with $\tilde{\tilde{M}}$ via the map $m \rightarrow \{f \rightarrow f(m)\}$. A G -morphism $\pi : M \rightarrow \tilde{M}$ will be called symmetric if the dual morphism $\tilde{\pi} : \tilde{\tilde{M}} = M \rightarrow \tilde{M}$ is equal to π . Thus π is symmetric if and only if the G -invariant bilinear form $\langle \cdot, \cdot \rangle_{\pi}$ defined by $\langle m, m' \rangle_{\pi} = \pi(m)(m')$ is symmetric, in the sense that $\langle m, m' \rangle_{\pi} = \langle m', m \rangle_{\pi}$ for all $m, m' \in M$. We will call π non-degenerate if it is an isomorphism, in which case $\langle \cdot, \cdot \rangle_{\pi}$ is a non-degenerate bilinear form.

Lemma 2.1 Define a biadditive form

$$\langle , \rangle_{\pi, G} : M \times M \mapsto \mathbf{Q}G$$

by

$$\langle m, m' \rangle_{\pi, G} = \sum_{g \in G} \langle m, gm' \rangle_{\pi} g^{-1}.$$

The form $\langle , \rangle_{\pi, G}$ is $\mathbf{Q}G$ -linear in the second variable. If π is symmetric then

$$\overline{\langle m, m' \rangle_{\pi, G}} = \langle m', m \rangle_{\pi, G}$$

where the anti-involution $\alpha \rightarrow \bar{\alpha}$ of $\mathbf{Q}G$ is the unique \mathbf{Q} -linear homomorphism for which $\bar{g} = g^{-1}$ for $g \in G$. Thus if M is a free finitely generated $\mathbf{Q}G$ -module and π is symmetric and non-degenerate, the form $\langle , \rangle_{\pi, G}$ is a non-degenerate Hermitian form on M in the sense of ([13], p. 25). Conversely, any such form is equal to $\langle , \rangle_{\pi, G}$ for a unique symmetric non-degenerate π .

Proof: See [13, p. 25, 164].

Suppose M^\bullet is a complex of finitely generated $\mathbf{Q}G$ -modules. Let \widetilde{M}^\bullet be the complex which results from applying the functor $M \rightarrow \widetilde{M}$ to M^\bullet . The i^{th} term of \widetilde{M}^\bullet is \widetilde{M}^{-i} .

Let $\phi : M^\bullet \rightarrow \widetilde{M}^\bullet$ be a morphism of complexes of finitely generated $\mathbf{Q}G$ -modules. We have a canonical identification of $\widetilde{\widetilde{M}^\bullet}$ with M^\bullet . Using this identification, let $\tilde{\phi} : M^\bullet = \widetilde{\widetilde{M}^\bullet} \rightarrow \widetilde{M}^\bullet$ be the morphism of complexes which results from applying the functor $M \mapsto \widetilde{M}$ to ϕ .

Definition 2.2 The morphism $\phi : M^\bullet \rightarrow \widetilde{M}^\bullet$ is symmetric if $\tilde{\phi} = \phi$. If for each $i \in \mathbf{Z}$, $\phi^i : M^i \rightarrow (\widetilde{M}^\bullet)^i = \widetilde{M}^{-i}$ is an isomorphism, we will call ϕ non-degenerate. If ϕ is both symmetric and non-degenerate, ϕ will be called perfect. A perfect Hermitian complex is a pair (P^\bullet, ϕ) consisting of a perfect complex P^\bullet of $\mathbf{Z}G$ modules and a perfect morphism $\phi : P_{\mathbf{Q}}^\bullet \rightarrow \widetilde{P_{\mathbf{Q}}^\bullet}$, where $P_{\mathbf{Q}}^\bullet = P^\bullet \otimes_{\mathbf{Z}} \mathbf{Q}$.

Definition 2.3 Suppose (P^\bullet, ϕ) is a perfect Hermitian complex. Define $P^{\text{even}} = \bigoplus_{i \text{ even}} P^i$ and $P^{\text{odd}} = \bigoplus_{i \text{ odd}} P^i$. We have G -homomorphisms

$$\phi^{\text{even}} = \sum_{i \text{ even}} \phi^i : P_{\mathbf{Q}}^{\text{even}} \rightarrow P_{\mathbf{Q}}^{\text{even}}$$

and

$$\phi^{\text{odd}} = \sum_{i \text{ odd}} \phi^i : P_{\mathbf{Q}}^{\text{odd}} \rightarrow P_{\mathbf{Q}}^{\text{odd}}.$$

These G -homomorphisms are symmetric and non-degenerate because $(\tilde{\phi})^i : P_{\mathbf{Q}}^i \rightarrow \widetilde{P_{\mathbf{Q}}^i}$ equals $\tilde{\phi}^{-i}$ and because ϕ^i is an isomorphism. Let h^{even} and h^{odd} be the Hermitian forms on $P_{\mathbf{Q}}^{\text{even}}$ and $P_{\mathbf{Q}}^{\text{odd}}$ associated to ϕ^{even} and ϕ^{odd} .

Definition 2.4 A Hermitian module [13, p. 25] is a pair (M, h) consisting of a locally free finitely generated $\mathbf{Z}G$ -module M together with a non-degenerate Hermitian pairing $h : M_{\mathbf{Q}} \times M_{\mathbf{Q}} \rightarrow \mathbf{Q}G$, where $M_{\mathbf{Q}} = M \otimes_{\mathbf{Z}} \mathbf{Q}$. Let $d(M, h)$ be the discriminant of (M, h) in the Hermitian classgroup $HCl(\mathbf{Z}G)$ (see [13, Theorem 3, p. 50], or §4 below). With the notations of Definition 2.3, let

$$d(P^{\bullet}, \phi) = d(P^{\text{even}}, h^{\text{even}}) / d(P^{\text{odd}}, h^{\text{odd}})$$

We will show in Appendix D the following result:

Theorem 2.5 The discriminant $d(P^{\bullet}, \phi)$ depends only on the homotopy class of the perfect morphism $\phi : P_{\mathbf{Q}}^{\bullet} \rightarrow \widetilde{P}_{\mathbf{Q}}^{\bullet}$.

Definition 2.6 Suppose P^{\bullet} is a perfect complex of $\mathbf{Z}G$ -modules. A perfect pairing \langle, \rangle on the cohomology $H^{\bullet}(P_{\mathbf{Q}}^{\bullet})$ is defined to be a collection $\{\langle, \rangle_t\}_t$ of perfect G -equivariant pairings

$$\langle, \rangle_t : H^t(P_{\mathbf{Q}}^{\bullet}) \times H^{-t}(P_{\mathbf{Q}}^{\bullet}) \rightarrow \mathbf{Q}$$

such that

$$\langle x, y \rangle_t = \langle y, x \rangle_{-t} \tag{2.1}$$

for all $t \in \mathbf{Z}$, $x \in H^t(P_{\mathbf{Q}}^{\bullet})$ and $y \in H^{-t}(P_{\mathbf{Q}}^{\bullet})$.

Proposition 2.7 Let P^{\bullet} be a perfect complex of $\mathbf{Z}G$ -modules.

- a. Suppose $\phi : P_{\mathbf{Q}}^{\bullet} \rightarrow \widetilde{P}_{\mathbf{Q}}^{\bullet}$ is a perfect morphism. Then ϕ induces isomorphisms $\phi^i : H^i(P_{\mathbf{Q}}^{\bullet}) \rightarrow H^i(\widetilde{P}_{\mathbf{Q}}^{\bullet}) = H^{-i}(P_{\mathbf{Q}}^{\bullet})$. The resulting pairings $\langle, \rangle_t : H^t(P_{\mathbf{Q}}^{\bullet}) \times H^{-t}(P_{\mathbf{Q}}^{\bullet}) \rightarrow \mathbf{Q}$ define a perfect pairing \langle, \rangle on $H^{\bullet}(P_{\mathbf{Q}}^{\bullet})$.
- b. Conversely, suppose $\langle, \rangle = \{\langle, \rangle_t\}_t$ is a perfect pairing on $H^{\bullet}(P_{\mathbf{Q}}^{\bullet})$. There is an acyclic perfect complex K^{\bullet} of $\mathbf{Z}G$ -modules so that the following is true of the complex $S^{\bullet} = P^{\bullet} \oplus K^{\bullet}$: there is a perfect morphism $\phi = \phi(\langle, \rangle) : S_{\mathbf{Q}}^{\bullet} \rightarrow \widetilde{S}_{\mathbf{Q}}^{\bullet}$ which induces the pairings \langle, \rangle_t via the natural isomorphisms $H^i(S_{\mathbf{Q}}^{\bullet}) \rightarrow H^i(P_{\mathbf{Q}}^{\bullet})$ for all i . The morphism ϕ is unique up to homotopy.

Proof: The only thing to show regarding part (a) is that the symmetry condition (2.1) holds, and this is a consequence of the fact that $\phi^t = \widetilde{\phi}^{-t}$ by assumption. We now assume the hypotheses of part (b).

By summing complexes of the form $F^m \xrightarrow{z} F^{m+1}$ to P^{\bullet} , in which F^j is a finitely generated free module in degree j and z is an isomorphism, we can construct an acyclic bounded complex K^{\bullet} of finitely generated free $\mathbf{Z}G$ -modules such that when $S^{\bullet} = P^{\bullet} \oplus K^{\bullet}$, the ranks of S^i and S^{-i} as locally free $\mathbf{Z}G$ -modules are equal if $i \neq 0$. This is also true if $i = 0$.

For each i , choose a non-canonical splitting

$$S_{\mathbf{Q}}^i = B^i \oplus H^i \oplus U^i$$

in which $H^i = H^i(S_{\mathbf{Q}}^{\bullet})$ and U^i is a $\mathbf{Q}G$ -module which maps isomorphically under the boundary homomorphism $\delta^i : S_{\mathbf{Q}}^i \rightarrow S_{\mathbf{Q}}^{i+1}$ to $B^i = B^{i+1}(S^{\bullet})$. We have a $\mathbf{Q}G$ -module isomorphism

$$S_{\mathbf{Q}}^i = \widetilde{S_{\mathbf{Q}}^{-i}} \quad (2.2)$$

since S^i and S^{-i} are locally free of the same rank as $\mathbf{Z}G$ -modules. Let

$$\phi_H^i : H^i \rightarrow \widetilde{H^{-i}} \quad (2.3)$$

be the isomorphism induced by \langle, \rangle_i . Clearly B^i and U^{-i} are $\{0\}$ if i is sufficiently negative. We conclude by induction, using (2.2) and (2.3), that for all $i \leq 0$ there is a $\mathbf{Q}G$ -module isomorphism

$$\phi_B^i : B^i \rightarrow \widetilde{U^{-i}}.$$

The boundary isomorphisms $U^{i-1} \rightarrow B^i$ and $U^{-i} \rightarrow B^{-i+1}$ together with ϕ_B^i induce isomorphisms

$$\phi_U^{i-1} : U^{i-1} \rightarrow \widetilde{B^{-i+1}}$$

for $i \leq 0$. Define $\phi_U^i : U^i \rightarrow \widetilde{B^i}$ to be $\widetilde{\phi_B^{-i}}$ for $i \geq 0$, and define $\phi_B^i : B^i \rightarrow \widetilde{U^{-i}}$ to be $\widetilde{\phi_U^{-i}}$ for $i > 0$. Define ϕ^i by

$$\phi^i = \phi_B^i \oplus \phi_H^i \oplus \phi_U^i : S_{\mathbf{Q}}^i = B^i \oplus H^i \oplus U^i \rightarrow \widetilde{S_{\mathbf{Q}}^{-i}} = \widetilde{U^{-i}} \oplus \widetilde{H^{-i}} \oplus \widetilde{B^{-i}}.$$

By construction, the ϕ^i fit together to give a morphism of complexes $\phi : S_{\mathbf{Q}}^{\bullet} \rightarrow \widetilde{S_{\mathbf{Q}}^{\bullet}}$. Furthermore, $\phi_B^i = \widetilde{\phi_U^{-i}}$ and $\phi_U^i = \widetilde{\phi_B^{-i}}$ for all i . Property (2.1) implies $\phi_H^i = \widetilde{\phi_H^{-i}}$ for all i , so we conclude $\phi^i = \widetilde{\phi^{-i}}$, i.e. that ϕ is a symmetric morphism in the sense of Definition 2.2. Since each ϕ^i was constructed to be an isomorphism, we also see that ϕ is non-degenerate, so ϕ is perfect. Because $\mathbf{Q}G$ is semi-simple, two morphisms from $S_{\mathbf{Q}}^{\bullet}$ to $\widetilde{S_{\mathbf{Q}}^{\bullet}}$ which induce the same morphism on cohomology must be homotopic. Hence ϕ is unique up to homotopy.

Theorem 2.8 *Suppose P^{\bullet} is a perfect complex of $\mathbf{Z}G$ -modules, and that $\langle, \rangle = \{\langle, \rangle_t\}_t$ is a perfect pairing on the cohomology $H^{\bullet}(P_{\mathbf{Q}}^{\bullet})$. We will say that two pairs $(P^{\bullet}, \langle, \rangle)$ and $(P'^{\bullet}, \langle, \rangle')$ of this kind are quasi-isomorphic if there is an isomorphism between P^{\bullet} and P'^{\bullet} in the derived category which identifies \langle, \rangle with \langle, \rangle' . Let $K^{\bullet}, S^{\bullet} = P^{\bullet} \oplus K^{\bullet}$ and $\phi = \phi(\langle, \rangle) : S_{\mathbf{Q}}^{\bullet} \rightarrow \widetilde{S_{\mathbf{Q}}^{\bullet}}$ be as in Proposition 2.7(b). The discriminant $d(S^{\bullet}, \phi)$ depends only on the quasi-isomorphism class of $(P^{\bullet}, \langle, \rangle)$, and will be denoted $d(P^{\bullet}, \langle, \rangle)$.*

Proof: Suppose that $S'^{\bullet} = P'^{\bullet} \oplus K'^{\bullet}$ and $\phi' : S'_{\mathbf{Q}}^{\bullet} \rightarrow \widetilde{S'_{\mathbf{Q}}^{\bullet}}$ are constructed for P'^{\bullet} in the same way S^{\bullet} and ϕ were for P^{\bullet} . There is then an isomorphism $\lambda : S^{\bullet} \rightarrow S'^{\bullet}$ in the derived category which identifies the cohomology groups of these complexes and the given pairings on these cohomology groups after they are tensored with \mathbf{Q} . Since S^{\bullet} is a complex of projective $\mathbf{Z}G$ -modules, we have from [20, Lemma VI.8.17] that we can take λ to be a morphism of complexes. Suppose now that T is a finitely generated free $\mathbf{Z}G$ -module. We add to S^{\bullet} complexes $F^i \xrightarrow{\eta} F^{i+1}$ and $F^{-i-1} \xrightarrow{\eta} F^{-i}$

in which $i < 0$, F^i and F^{i+1} are isomorphic to T and η is an isomorphism, and F^{-i} and F^{-i-1} are isomorphic to $\hat{T} = \text{Hom}_{\mathbf{Z}}(T, \mathbf{Z})$ and $\hat{\eta}$ is the \mathbf{Z} -dual of η . We extend ϕ to these complexes summed to S^\bullet using the canonical isomorphisms $F^i \rightarrow \hat{F}^{-i}$ and $F^{i+1} \rightarrow \hat{F}^{-i-1}$. This does not change $d(S^\bullet, \phi)$, and enables us to extend $\lambda : S^\bullet \rightarrow S'^\bullet$ in such a way that each homomorphism $S^i \rightarrow S'^i$ is surjective. Again using [20, Lemma VI.8.17], we get a morphism of complexes $\tau : S'^\bullet \rightarrow S^\bullet$ which is a section of λ . Thus if $T^\bullet = \ker(\lambda)$, then we have a morphism $T^\bullet \oplus S'^\bullet \rightarrow S^\bullet$ which is a term-by-term isomorphism. Furthermore, T^\bullet is acyclic and perfect, since λ is a quasi-isomorphism. Note that since S^i and S^{-i} have the same rank as locally $\mathbf{Z}G$ -modules, and the same is true of S'^i and S'^{-i} , we know that T^i and T^{-i} have the same rank as locally free $\mathbf{Z}G$ -modules.

Now T^\bullet is an acyclic perfect complex. By increasing induction on i , we find that $T^i = B^i(T^\bullet) \oplus U_{\mathbf{Z}}^i$, where $B^i(T^\bullet)$ and $U_{\mathbf{Z}}^i(T^\bullet)$ are projective $\mathbf{Z}G$ -modules, and the boundary homomorphism gives an isomorphism between $U_{\mathbf{Z}}^i(T^\bullet)$ and $B_{\mathbf{Z}}^{i+1}$. We now employ the construction of the proof of Proposition 2.7 to construct a perfect morphism $\phi_T : T_{\mathbf{Q}}^\bullet \rightarrow \widetilde{T}_{\mathbf{Q}}^\bullet$. With the notations of Proposition 2.7, we find that for $i \neq 0$, $T^i \oplus T^{-i}$ is the perpendicular sum of $B^i(T^\bullet) \oplus U_{\mathbf{Z}}^{-i}(T^\bullet)$ and $U_{\mathbf{Z}}^i(T^\bullet) \oplus B^{-i}(T^\bullet)$ with respect to the Hermitian form induced by ϕ . Furthermore, $B^i(T^\bullet) \oplus U_{\mathbf{Z}}^{-i}$ is isogenous to a hyperbolic plane of the form $A \oplus \hat{A}$, where A is a free $\mathbf{Z}G$ -module contained in $B^i(T^\bullet)$ with finite index. The discriminants of such hyperbolic planes are equal to 1 by [13, p. 53]. By using the isomorphisms $U_{\mathbf{Z}}^{i-1} \rightarrow B^i(T^\bullet)$ and $U_{\mathbf{Z}}^{-i} \rightarrow B_{\mathbf{Z}}^{-i+1}$, we can compare the hyperbolic planes and isogenies corresponding to $U_{\mathbf{Z}}^i \oplus B^{-i}(T^\bullet)$ and $B^{i+1}(T^\bullet) \oplus U_{\mathbf{Z}}^{-i-1}$. Taking into account also the case $i = 0$, we see that the factors associated to the isogenies cancel one another in the formula

$$d(T^\bullet, \phi_T) = d(T^0, h) \cdot \prod_{i < 0} d(T^i \oplus T^{-i}, h)^{(-1)^i}$$

where on the right, h stands for the Hermitian form induced by ϕ_T on the appropriate module. This, and the fact that hyperbolic planes have discriminant 1, shows

$$d(T^\bullet, \phi_T) = 1.$$

The sum $\phi_T \oplus \phi'$ is a perfect morphism from $T_{\mathbf{Q}}^\bullet \oplus S'_{\mathbf{Q}}^\bullet = S_{\mathbf{Q}}^\bullet$ to $\widetilde{S}_{\mathbf{Q}}^\bullet$ which induces the same cohomology pairings as ϕ' , and hence also the same cohomology pairings as ϕ . Thus $\phi_T \oplus \phi'$ must be homotopic to ϕ , so

$$d(S^\bullet, \phi_T \oplus \phi') = d(S^\bullet, \phi)$$

by Theorem 2.5. But clearly

$$d(S^\bullet, \phi_T \oplus \phi') = d(T^\bullet \oplus S'^\bullet, \phi_T \oplus \phi') = d(T^\bullet, \phi_T) \cdot d(S'^\bullet, \phi') = d(S'^\bullet, \phi').$$

Combining these last two equalities shows $d(S^\bullet, \phi) = d(S'^\bullet, \phi')$, so Theorem 2.8 is proved.

3 Hermitian de Rham discriminants.

In this section we describe an arithmetic application of the results of §2.

Let \mathcal{W} be a regular scheme which is projective over a Noetherian ring A . Suppose the finite group G acts on \mathcal{W} over A . By an $O_{\mathcal{W}}-G$ module we will mean a coherent $O_{\mathcal{W}}$ -module having an action of

G compatible with the action of G on $\mathcal{O}_{\mathcal{W}}$. Define $G_0(G, \mathcal{W})$ (resp. $K_0(G, \mathcal{W})$) to be the Grothendieck group of the category $\text{Coh}(G, \mathcal{W})$ (resp. $\mathcal{P}(G, \mathcal{W})$) of $\mathcal{O}_{\mathcal{W}}$ - G modules (resp. $\mathcal{O}_{\mathcal{W}}$ - G which are locally free as $\mathcal{O}_{\mathcal{W}}$ -modules). For each $i \geq 0$, one has a lambda operator $\lambda^i : K_0(G, \mathcal{W}) \rightarrow K_0(G, \mathcal{W})$ which sends the class of a vector bundle \mathcal{E} to the class of $\Lambda^i_{\mathcal{O}_{\mathcal{W}}} \mathcal{E}$. Since \mathcal{W} is regular, the forgetful homomorphism $K_0(G, \mathcal{W}) \rightarrow G_0(G, \mathcal{W})$ is an isomorphism by [27, §1.3, Remark 1.9(a)], and using this homomorphism we will identify these two groups. We will let $D(G, \mathcal{W})$ be the derived category of the homotopy category of bounded complexes L^\bullet of $\mathcal{O}_{\mathcal{W}}$ - G -modules.

If G is a finite group acting tamely on \mathcal{W} , in the sense of [5], then it follows from the results of [5, §8] that the hypercohomology $R, (\mathcal{W}, L^\bullet)$ is represented by a bounded complex of finitely generated AG -modules which are cohomologically trivial for G . There is then an Euler characteristic $\chi_A(R, (\mathcal{W}, L^\bullet))$ in the Grothendieck group $CT(AG)$ of all such AG -modules. If A has finite projective dimension (e.g. if A is a field or a Dedekind ring), we can (and will) take $R, (\mathcal{W}, L^\bullet)$ to be a perfect complex of AG -modules, and $\chi_A(R, (\mathcal{W}, L^\bullet))$ to be an element of $K_0(AG)$. When there is no danger of confusion, we will write $R, (L^\bullet)$ for $R, (\mathcal{W}, L^\bullet)$ and $H^j(L^\bullet)$ for $H^j(\mathcal{W}, L^\bullet)$.

Suppose now that \mathcal{X} is a regular projective scheme which is flat over \mathbf{Z} and equidimensional of dimension $d + 1$. Let G be a finite group acting tamely on \mathcal{X} , in the sense of [5]. Choose a G -equivariant embedding $\mathcal{X} \rightarrow \mathcal{P}$ of \mathcal{X} into a scheme \mathcal{P} which is smooth and projective over \mathbf{Z} , and let I be the ideal sheaf of this embedding. This gives a G -equivariant exact sequence

$$0 \rightarrow T \rightarrow E \rightarrow \Omega_{\mathcal{X}/\mathbf{Z}}^1 \rightarrow 0.$$

in which $T = I/I^2$ and $E = \Omega_{\mathcal{P}/\mathbf{Z}}^1|_{\mathcal{X}}$ are coherent locally free $\mathcal{O}_{\mathcal{X}}$ - G -modules. The above exact sequence gives a quasi-isomorphism between the complex $K^\bullet = [T \rightarrow E]$ having T in degree -1 and E in degree 0 and the complex $\Omega_{\mathcal{X}/\mathbf{Z}}^1[0]$ which has $\Omega_{\mathcal{X}/\mathbf{Z}}^1$ in degree 0 .

For $p \geq 0$ we will denote by $L \wedge^p$ the p -th left derived exterior power functor as defined on perfect complexes of $\mathcal{O}_{\mathcal{X}}$ -modules by Dold-Puppe [10] (see also Illusie [15] and [21, §5.4 - 5.9]). The construction of $L \wedge^p K^\bullet$ for all $K^\bullet = [T \rightarrow E]$ supported at -1 and 0 is described in [23, §1.1]. We now recall this construction.

We first define a p -complex as follows: If $a = (a_1, \dots, a_p) \in \mathbf{Z}^p$ we set $|a| = \sum_{i=1}^p a_i$ and $|a'| = |a| - a_p$. If $a_i = 0$ or 1 for all $1 \leq i \leq p$, let $m_1 < \dots < m_{|a'|}$ be the integers $1 \leq m < p$ such that $a_m = 1$ and put $n_i = m_i - m_{i-1}$ for $1 \leq i \leq |a'| + 1$, where $m_0 = 0$ and $m_{|a'|+1} = p$. The a component of the p -complex is

$$(L \wedge^p K^\bullet)_a = \begin{cases} \wedge^{n_1} T \otimes \dots \otimes \wedge^{n_{|a'|+1}} T, & \text{if } a_i = 0 \text{ or } 1 \text{ for } 1 \leq i < p \text{ and } a_p = 1 \\ \wedge^{n_1} T \otimes \dots \otimes \wedge^{n_{|a'|}} T \otimes \wedge^{n_{|a'|+1}} E, & \text{if } a_i = 0 \text{ or } 1 \text{ for } 1 \leq i < p \text{ and } a_p = 0 \\ 0, & \text{otherwise.} \end{cases}$$

The boundary morphisms are all given by natural morphisms obtained from $\wedge^n T \otimes \wedge^m T \rightarrow \wedge^{n+m} T$, $\wedge^n T \otimes \wedge^m E \rightarrow \wedge^{n+m} E$ etc. Notice that for all a we have $n_1 + \dots + n_{|a'|+1} = p$.

The complex $L \wedge^p K^\bullet$ is now defined to be the associated simple complex. Its terms are:

$$(L \wedge^p K^\bullet)^q = \bigoplus_{a \in \mathbf{Z}^p, |a| = -q} (L \wedge^p K^\bullet)_a.$$

For example, if $p = 1$, $L \wedge^1 K^\bullet = K^\bullet$; if $p = 2$, $L \wedge^2 K^\bullet$ is the complex

$$T \otimes T \longrightarrow (\wedge^2 T) \oplus (T \otimes E) \longrightarrow \wedge^2 E.$$

at $-2, -1, 0$. All the morphisms are natural and therefore G -equivariant.

The complex $L \wedge^p K^\bullet$ is a bounded complex of coherent locally free $\mathcal{O}_{\mathcal{X}}$ - G -modules. If F^\bullet is a bounded complex of coherent $\mathcal{O}_{\mathcal{X}}$ - G -modules, set $H(F^\bullet) = \sum_i (-1)^i [F^i] \in G_0(G, \mathcal{X})$.

Proposition 3.1 *The isomorphism class of $L \wedge^p K^\bullet$ in the derived category $D(G, \mathcal{X})$ is independent of the choice of \mathcal{P} and of the G -equivariant embedding $\mathcal{X} \rightarrow \mathcal{P}$. There is an isomorphism in $D(G, X)$ between the restriction of $L \wedge^p K^\bullet$ to the general fiber $X = \mathcal{X}_{\mathbf{Q}}$ of \mathcal{X} and $\Omega_{X/\mathbf{Q}}^p[0]$. In $G_0(G, \mathcal{X}) = K_0(G, \mathcal{X})$ one has $H(L \wedge^p K^\bullet) = \lambda^p(\Omega_{\mathcal{X}/\mathbf{Z}}^1)$.*

Proof: Suppose that $\mathcal{X} \rightarrow \mathcal{P}'$ is a second G -equivariant embedding which produces the complex $K'^\bullet = [T' \rightarrow E']$. Consider the product embedding $\mathcal{X} \rightarrow \mathcal{P}'' := \mathcal{P} \times \mathcal{P}'$ which gives $K''^\bullet := [T'' \rightarrow E'']$. We have

$$\Omega_{\mathcal{P} \times \mathcal{P}'/\mathbf{Z}}^1 = p^* \Omega_{\mathcal{P}/\mathbf{Z}}^1 \oplus p'^* \Omega_{\mathcal{P}'/\mathbf{Z}}^1$$

and so $E'' = E \oplus E'$. There are G -morphisms of complexes $K''^\bullet \rightarrow K^\bullet$ and $K''^\bullet \rightarrow K'^\bullet$ which are quasi-isomorphisms. The corresponding morphisms $L \wedge^p K''^\bullet \rightarrow L \wedge^p K^\bullet$ and $L \wedge^p K''^\bullet \rightarrow L \wedge^p K'^\bullet$ are G -equivariant. Since $L \wedge^p$ is a functor on the derived category of $\mathcal{O}_{\mathcal{X}}$ -modules they are also quasi-isomorphisms. (This is also explained in [23, Lemma (1.2), Corollary].) Hence $L \wedge^p K^\bullet$ is isomorphic to $L \wedge^p K'^\bullet$ in $D(G, \mathcal{X})$. Because $X = \mathcal{X}_{\mathbf{Q}}$ is smooth over \mathbf{Q} , we can use the identity map on X as a G -equivariant embedding of X into a smooth scheme over \mathbf{Q} . The above reasoning applied over X now leads to an isomorphism in $D(G, X)$ between the restriction of $L \wedge^p K^\bullet$ and $\Omega_{X/\mathbf{Q}}^p[0]$.

To prove $H(L \wedge^p K^\bullet) = \lambda^p(\Omega_{\mathcal{X}/\mathbf{Z}}^1)$ by induction on p , note first that this is true if $p = 0$, when $H(L \wedge^0 K^\bullet) = 1 = \lambda^0(\Omega_{\mathcal{X}/\mathbf{Z}}^1)$, and if $p = 1$, when $H(L \wedge^1 K^\bullet) = E - T = \Omega_{\mathcal{X}/\mathbf{Z}}^1$. Suppose now that $p > 1$. If $n_1 + \dots + n_l = p$, $n_i \geq 1$, then $P = (n_1, \dots, n_l)$ is a partition of p of length l . By the explicit description of $L \wedge^p K^\bullet$ given above we have:

$$H(L \wedge^p K^\bullet) = \sum_P (-1)^{l(P)} (\wedge^{n_1} T \otimes \dots \otimes \wedge^{n_l(P)} T - \wedge^{n_1} T \otimes \dots \otimes \wedge^{n_l(P)-1} T \otimes \wedge^{n_l(P)} E)$$

where the sum is over all partitions P of p ; when $l(P) = 1$ the second summand in the parenthesis is equal to $\wedge^{n_l(P)} E$. It follows that we have

$$H(L \wedge^p K^\bullet) = \wedge^p E - \sum_{r+s=p, r \geq 1} \wedge^r T \otimes H(L \wedge^s K^\bullet). \quad (3.1)$$

On the other hand, by $\Omega_{\mathcal{X}/\mathbf{Z}}^1 = E - T$ and standard properties of the lambda operators we have

$$\lambda^p(\Omega_{\mathcal{X}/\mathbf{Z}}^1) = \wedge^p E - \sum_{r+s=p, r \geq 1} \wedge^r T \otimes \lambda^s(\Omega_{\mathcal{X}/\mathbf{Z}}^1) \quad (3.2)$$

where both equalities are in $K_0(G, \mathcal{X}) = G_0(G, \mathcal{X})$. The Lemma follows by (3.1), (3.2) and the induction hypothesis.

Let $M^\bullet[m]$ be image of a complex M^\bullet under the m^{th} power of the shift operator ([19], p. 26), so that $M^\bullet[m]$ has j^{th} term M^{j+m} .

Corollary 3.2 *Define $\lambda^{\leq d}(\Omega_{\mathcal{X}/\mathbf{Z}}^1)$ to be the direct sum for $0 \leq p \leq d$ of the complexes $L \wedge^p K^\bullet[-p]$. The isomorphism class of $L \wedge^p K^\bullet$ in $D(G, \mathcal{X})$ is independent of the choice of \mathcal{P} and of the embedding $\mathcal{X} \rightarrow \mathcal{P}$. The restriction $\lambda^{\leq d}(\Omega_{\mathcal{X}/\mathbf{Z}}^1)_{\mathbf{Q}}$ of $\lambda^{\leq d}(\Omega_{\mathcal{X}/\mathbf{Z}}^1)$ to $X = \mathcal{X}_{\mathbf{Q}}$ is isomorphic in $D(G, X)$ to the complex having $\Omega_{X/\mathbf{Q}}^p$ in degree p for all p and trivial boundary homomorphisms. One has $H(\lambda^{\leq d}(\Omega_{\mathcal{X}/\mathbf{Z}}^1)) = \sum_{p=0}^d (-1)^p \cdot \lambda^p(\Omega_{\mathcal{X}/\mathbf{Z}}^1)$.*

As before, let $R, (\lambda^{\leq d}(\Omega_{\mathcal{X}/\mathbf{Z}}^1)[d])$ be a perfect complex of $\mathbf{Z}G$ -modules representing the hypercohomology of $\lambda^{\leq d}(\Omega_{\mathcal{X}/\mathbf{Z}}^1)[d]$ over \mathcal{X} . Corollary 3.2 and flat base change from \mathcal{X} to X give canonical isomorphisms

$$\begin{aligned} H^t(R, (\lambda^{\leq d}(\Omega_{\mathcal{X}/\mathbf{Z}}^1)[d])_{\mathbf{Q}}) &= H^t(R, (\lambda^{\leq d}(\Omega_{\mathcal{X}/\mathbf{Z}}^1)_{\mathbf{Q}}[d])) \\ &= H^t(R, (\bigoplus_{p=0}^d \Omega_{X/\mathbf{Q}}^p[d-p])) \\ &= \bigoplus_{i+p=t+d} H^i(X, \Omega_X^p) \end{aligned} \quad (3.3)$$

for all t , where $R, (\lambda^{\leq d}(\Omega_{\mathcal{X}/\mathbf{Z}}^1)[d])_{\mathbf{Q}} = R, (\lambda^{\leq d}(\Omega_{\mathcal{X}/\mathbf{Z}}^1)[d]) \otimes_{\mathbf{Z}} \mathbf{Q}$, and the left and right sides of (3.3) are 0 if $t < -d$ or $t > d$.

From [18, III,§7] we have canonical perfect G -equivariant duality pairings

$$\langle, \rangle_{i,j} : H^i(X, \Omega_X^j) \times H^{d-i}(X, \Omega_X^{d-j}) \rightarrow \mathbf{Q}$$

for $0 \leq i, j \leq d$. Define

$$\langle, \rangle'_{i,j} : H^i(X, \Omega_X^j) \times H^{d-i}(X, \Omega_X^{d-j}) \rightarrow \mathbf{Q}$$

by

$$\langle x, y \rangle'_{i,j} = \langle y, x \rangle_{j,i}. \quad (3.4)$$

By comparing $\langle, \rangle_{i,j}$ to the intersection pairing on Betti-cohomology (c.f. [14]), we see that

$$\langle x, y \rangle'_{i,j} = (-1)^{(i+j)} \langle x, y \rangle_{j,i} \quad (3.5)$$

since $i + j$ and $2d - (i + j)$ have the same parity.

Definition 3.3 *Define a G -equivariant non-degenerate pairing*

$$\langle, \rangle_t : H^t(R, (\lambda^{\leq d}(\Omega_{\mathcal{X}/\mathbf{Z}}^1)[d])_{\mathbf{Q}}) \times H^{-t}(R, (\lambda^{\leq d}(\Omega_{\mathcal{X}/\mathbf{Z}}^1)[d])_{\mathbf{Q}}) \rightarrow \mathbf{Q}$$

in the following way: if $t < 0$, let

$$\langle , \rangle_t = \bigoplus_{i+j=t+d} \langle , \rangle_{i,j}$$

relative to the canonical direct sum decomposition in (3.3) for t and $-t$. If $t > 0$, define

$$\langle x, y \rangle_t = \langle y, x \rangle_{-t} = \bigoplus_{i+j=t+d} \langle , \rangle'_{i,j} = (-1)^{t+d} \cdot \bigoplus_{i+j=t+d} \langle , \rangle_{i,j}.$$

Finally, if $t = 0$, let

$$\langle , \rangle_0 = \bigoplus_{i < d/2} \langle , \rangle_{i,d-i} \oplus \langle , \rangle_{d/2,d/2} \oplus \bigoplus_{i > d/2} \langle , \rangle'_{i,d-i}.$$

Here the term $\langle , \rangle_{d/2,d/2}$ appears only if d is even, and in this case $\langle , \rangle_{d/2,d/2} = \langle , \rangle'_{d/2,d/2}$ is a symmetric pairing on $H^{d/2}(X, \Omega_X^{d/2})$ because of (3.5). Thus

$$\langle x, y \rangle_t = \langle y, x \rangle_{-t}$$

for all t , $x \in H^t(R, (\lambda^{\leq d}(\Omega_{\mathcal{X}/\mathbf{Z}}^1)[d])_{\mathbf{Q}})$ and $y \in H^{-t}(R, (\lambda^{\leq d}(\Omega_{\mathcal{X}/\mathbf{Z}}^1)[d])_{\mathbf{Q}})$.

Theorem 3.4 *There is a perfect Hermitian complexes (P^\bullet, ϕ) with the following properties:*

- P^\bullet is isomorphic to $R, (\lambda^{\leq d}(\Omega_{\mathcal{X}/\mathbf{Z}}^1)[d])$ in the derived category of the homotopy category of bounded complexes $\mathbf{Z}G$ -modules.
- Identify $H^t(P_{\mathbf{Q}}^\bullet)$ with $H^t(R, (\lambda^{\leq d}(\Omega_{\mathcal{X}/\mathbf{Z}}^1)[d])_{\mathbf{Q}})$ via the isomorphism in part (a). The isomorphism $\phi : P_{\mathbf{Q}}^\bullet \rightarrow \widetilde{P_{\mathbf{Q}}^\bullet}$ induces on cohomology the pairing $\langle , \rangle_t : H^t(P_{\mathbf{Q}}^\bullet) \times H^{-t}(P_{\mathbf{Q}}^\bullet) \rightarrow \mathbf{Q}$ of Definition 3.3 for all t . This property determines ϕ uniquely up to homotopy.

Proof: By [5], $R, (\lambda^{\leq d}(\Omega_{\mathcal{X}/\mathbf{Z}}^1)[d])$ and $R, (\Omega_{\mathcal{X}/\mathbf{Z}}^\bullet)$ are isomorphic to perfect complexes of $\mathbf{Z}G$ -modules in the derived category. Theorem 3.4 now follows from Proposition 2.7(b)

Corollary 3.5 *The discriminant $d(P^\bullet, \phi)$ depends only on \mathcal{X} and the action of G on \mathcal{X} , and will be called the Hermitian de Rham invariant $\chi_H(\mathcal{X}, G) \in HCl(\mathbf{Z}G)$ of this G action. The image of $\chi_H(\mathcal{X}, G)$ in $Cl(\mathbf{Z}G)$ is $(-1)^d$ times the de Rham invariant*

$$\chi(\mathcal{X}, G) = \sum_i (-1)^i \cdot \chi^P(\lambda^i(\Omega_{\mathcal{X}/\mathbf{Z}}^1))$$

considered in [4].

Proof: This follows from Theorem 2.8.

4 Discriminants of Hermitian modules

We recall in §4 and §5 some definitions and results concerning Hermitian class groups. These are needed in order to state in §6 two results relating Hermitian de Rham discriminants to ϵ -factors.

It will be useful later to work over an arbitrary number field F . Let O_F be the ring of integers of F . Suppose M is a finitely generated locally free $O_F G$ -module. Define $M_F = F \otimes_{O_F} M$. Let h_F be a non-degenerate Hermitian pairing $h_F : M_F \times M_F \rightarrow FG$. Thus

$$h_F(v_2, v_1) = \overline{h_F(v_1, v_2)} \quad \text{and} \quad h_F(v_1, \alpha v_2) = \alpha \cdot h_F(v_1, v_2) \quad (4.1)$$

for $v_1, v_2 \in M_F$ and $\alpha = \sum_{g \in G} a_g g \in FG$, where $\overline{\alpha} = \sum_{g \in G} a_g g^{-1}$. The pair (M, h_F) is called a Hermitian module. (We should point out that in [13], all modules are right modules; the fact that we work with left modules will not cause difficulty.)

Fix an embedding of F into $\overline{\mathbf{Q}}$. A representation $T : G \rightarrow GL_n(\overline{\mathbf{Q}})$ is symplectic if it leaves invariant an alternating non-degenerate bilinear form $\beta : \overline{\mathbf{Q}}^n \times \overline{\mathbf{Q}}^n \rightarrow \overline{\mathbf{Q}}$, in the sense that $\beta(T(g)x, T(g)y) = \beta(x, y)$ for all $g \in G$ and $x, y \in \overline{\mathbf{Q}}^n$. We will let T also stand for the induced algebra homomorphism $FG \rightarrow Mat_n(\overline{\mathbf{Q}})$. The adjoint P^J of a matrix $P \in Mat_n(\overline{\mathbf{Q}})$ with respect to β is defined by requiring $\beta(P^J x, y) = \beta(x, Py)$ for all $x, y \in \overline{\mathbf{Q}}^n$. Thus T leaves β invariant if and only if

$$T(\alpha)^J = T(\overline{\alpha}) \quad (4.2)$$

for $\alpha \in FG$.

Let $\{v_i\}_{i=1}^q$ be a basis over FG of the free FG -module M_F . We then have a matrix $\{h_F(v_i, v_j)\}_{i,j=1}^q$ in $Mat_q(FG)$. Extend T to an algebra homomorphism $Mat_q(FG) \rightarrow Mat_{nq}(\overline{\mathbf{Q}})$ in the natural way. A matrix in $Mat_{nq}(\overline{\mathbf{Q}})$ may be viewed as a block matrix $B = \{b_{r,s}\}_{r,s=1}^q$ with entries $b_{r,s} \in Mat_n(\overline{\mathbf{Q}})$. We extend the adjoint map $P \rightarrow P^J$ to $Mat_{nq}(\overline{\mathbf{Q}})$ by letting $B^J = \{b_{s,r}^J\}_{r,s=1}^q$. This extension corresponds to the adjoint for the bilinear form on the orthogonal sum of q copies of $(\overline{\mathbf{Q}}^n, \beta)$.

A matrix $S \in Mat_{nq}(\overline{\mathbf{Q}})$ is called J -symmetric if $S^J = S$. The matrix $T(H) = \{T(h_F(v_i, v_j))\}_{i,j=1}^q$ is a J -symmetric element of $Mat_{nq}(\overline{\mathbf{Q}})$ because (4.1) and (4.2) imply

$$\{T(h_F(v_j, v_i))^J\}_{i,j=1}^q = \{T(\overline{h_F(v_i, v_j)})^J\}_{i,j=1}^q = \{T(h_F(v_i, v_j))\}_{i,j=1}^q.$$

Proposition 4.1 *If S is a J -symmetric element of $Mat_{nq}(\overline{\mathbf{Q}})$ there exists a matrix $P \in Mat_{nq}(\overline{\mathbf{Q}})$ such that $P^J P = S$. The determinant $Pf^J(S) = \det(P)$ depends only on S , and has the following properties:*

- a. $\det(S) = Pf^J(S)^2$.
- b. If $R \in GL_{nq}(\overline{\mathbf{Q}})$ then $R^J S R$ is J -symmetric and $Pf^J(R^J S R) = \det(R) Pf^J(S)$.

Proof: See [13, p. 28 - 29].

Proposition 4.2 *Suppose $\alpha = \{\alpha_{i,j}\}_{i,j} \in GL_q(\overline{\mathbf{Q}}G)$. Define $\overline{\alpha} = \{\overline{\alpha}_{j,i}\}_{i,j}$. If $\overline{\alpha} = \alpha$ the Pfaffian $Pf^J(T(\alpha))$ depends only on the character χ of the symplectic representation T . Hence we may*

define $Pf_\chi(\alpha) = Pf^J(T(\alpha))$. The homomorphism $Pf(\alpha) : \chi \rightarrow Pf_\chi(\alpha)$ defines an element of $\text{Hom}_{\Omega_F}(R_G^s, \overline{\mathbf{Q}}^*)$, where $\Omega_F = \text{Gal}(\overline{\mathbf{Q}}/F)$ acts on the Grothendieck group R_G^s of symplectic $\overline{\mathbf{Q}}$ -valued virtual characters of G via its action on $\overline{\mathbf{Q}}$. The homomorphism $Pf : \alpha \rightarrow Pf(\alpha)$ sends the symmetric elements of $GL_q(FG)$ to $\text{Hom}_{\Omega_F}(R_G^s, \overline{\mathbf{Q}}^*)$.

Proof: See [13, Prop. 4.2 and Prop. 4.3].

Definition 4.3 Let $J(FG)$ (resp. $U(O_F G)$) be the idele group (resp. the group of integral unit ideles) of FG . Let R_G be the group of $\overline{\mathbf{Q}}$ -valued virtual characters of G , and let $J(\overline{\mathbf{Q}})$ be the group of ideles of $\overline{\mathbf{Q}}$. We then have determinant homomorphisms $\text{Det} : J(FG) \rightarrow \text{Hom}_{\Omega_F}(R_G, J(\overline{\mathbf{Q}}))$ and $\text{Det} : FG^* \rightarrow \text{Hom}_{\Omega_F}(R_G, \overline{\mathbf{Q}}^*)$ as in ([13], p. 14 - 15). The Hermitian class group $HCl(O_F G)$ is defined to be the cokernel of the homomorphism

$$\Delta : \text{Det}(FG^*) \rightarrow \frac{\text{Det}(J(FG))}{\text{Det}(U(O_F G))} \times \text{Hom}_{\Omega_F}(R_G^s, \overline{\mathbf{Q}}^*)$$

defined by

$$\Delta(a) = (a^{-1} \bmod \text{Det}(U(O_F G)) \quad , \quad a^s)$$

where a^s is the restriction of a from R_G to R_G^s .

Definition 4.4 Suppose that the basis $\{v_i\}_{i=1}^q$ for the free FG -module $M_F = F \otimes_{O_F} M$ lies in M . Let $Ad(F)$ be the adèle ring of F . If v is a finite place of F , let O_v be the ring of integers of the completion F_v . If v an infinite place of F , let $O_v = F_v$. Because M is locally free, there is an element $\lambda = (\lambda_v)_v$ of $GL_q(Ad(F) \otimes_F FG)$ such that for all places v of F , multiplication by λ_v carries $\bigoplus_{i=1}^q O_v G v_i$ isomorphically onto $O_v \otimes_{O_F} M$. The discriminant $d(M, h_F)$ of the Hermitian module (M, h_F) is defined to be the class in $HCl(O_F G)$ given by

$$(\text{Det}(\lambda) \quad , \quad Pf((h_F(v_i, v_j))_{i,j}) \bmod \text{Image}(\Delta))$$

5 The Adelic Hermitian classgroup and rational classes

With the notations of the §4, let $\text{Det}^s(U(O_F G))$ be the subgroup of $\text{Hom}_{\Omega_F}(R_G^s, J(\overline{\mathbf{Q}}))$ formed by the restrictions to R_G^s of elements of $\text{Det}(U(O_F G))$. The adelic Hermitian classgroup $Ad HCl(O_F G)$ is defined by

$$Ad HCl(O_F G) = \frac{\text{Hom}_{\Omega_F}(R_G^s, J(\overline{\mathbf{Q}}))}{\text{Det}^s(U(O_F G))}.$$

There is a natural homomorphism

$$HCl(O_F G) \rightarrow Ad HCl(O_F G)$$

induced by the homomorphism

$$\frac{\text{Det}(J(FG))}{\text{Det}(U(O_F G))} \times \text{Hom}_{\Omega_F}(R_G^s, \overline{\mathbf{Q}}^*) \rightarrow \frac{\text{Hom}_{\Omega_F}(R_G^s, J(\overline{\mathbf{Q}}))}{\text{Det}^s(U(O_F G))}$$

defined by

$$(h \bmod \text{Det}(U(O_F G)), f) \rightarrow h^s \cdot f.$$

Define $(\pm 1)_f$ to be the idele in $J(F)$ whose finite components at ± 1 and whose infinite components are 1. Fix an embedding $\Delta_f : \mathbf{Q}^* = \{\pm 1\} \times \mathbf{Q}_{>0} \rightarrow J(\mathbf{Q})$ by requiring that $\Delta_f(-1) = (-1)_f$ and that $\Delta_f(r)$ have component r at all places of \mathbf{Q} (including the infinite one) if $0 < r \in \mathbf{Q}^*$. Note that Δ_f agrees with the usual diagonal embedding $\mathbf{Q}^* \rightarrow J(\mathbf{Q})$ only on the positive elements of \mathbf{Q}^* .

Proposition 5.1 (Cassou-Noguès - Taylor [13, p. 190]) *Suppose F is at most tamely ramified over the prime 2. In $\text{Hom}_{\Omega_F}(R_G^s, J(\overline{\mathbf{Q}}))$ one has*

$$\text{Hom}_{\Omega_F}(R_G^s, \{(\pm 1)_f\}) \cap \text{Det}^s(U(O_F G)) = \{1\}.$$

Therefore when $F = \mathbf{Q}$, the embedding Δ_f induces an embedding

$$\Delta_{f,*} : \text{Hom}_{\Omega_{\mathbf{Q}}}(R_G^s, \mathbf{Q}^*) \rightarrow \text{Ad } HCl(\mathbf{Z}G) = \frac{\text{Hom}_{\Omega_{\mathbf{Q}}}(R_G^s, J(\overline{\mathbf{Q}}))}{\text{Det}^s(U(\mathbf{Z}G))}.$$

Call the image of $\Delta_{f,*}$ the rational classes in $\text{Ad } HCl(\mathbf{Z}G)$.

Note that while the rational classes are naturally isomorphic to $\text{Hom}_{\Omega_{\mathbf{Q}}}(R_G^s, \mathbf{Q}^*)$, this isomorphism does not come about from the diagonal embedding of \mathbf{Q}^* into $J(\mathbf{Q})$ but rather from Δ_f .

Example 5.2 *Suppose $F = \mathbf{Q}$ and that G is the trivial group. The trivial character χ_0 (resp. $2 \cdot \chi_0$) generates R_G (resp. R_G^s) as an abelian group. The homomorphism Δ in Definition 4.3 is thus naturally identified with the homomorphism*

$$\Delta : \mathbf{Q}^* \rightarrow (\mathbf{Q}^*/\{\pm 1\}) \times \mathbf{Q}^* \tag{5.1}$$

defined by

$$\Delta(a) = (a^{-1} \bmod \{\pm 1\}, a^2)$$

where the diagonal inclusion $\mathbf{Q}^* \rightarrow J(\mathbf{Q})$ induces isomorphisms

$$\mathbf{Q}^*/\{\pm 1\} = J(\mathbf{Q})/U(\mathbf{Q}) = \text{Det}(J(\mathbf{Q}))/\text{Det}(U(\mathbf{Q}))$$

and evaluation on $2 \cdot \chi_0$ gives an isomorphism $\text{Hom}(R_G^s, \overline{\mathbf{Q}}^*) \rightarrow \mathbf{Q}^*$. From this we have an isomorphism

$$HCl(\mathbf{Z}) \rightarrow \mathbf{Q}^* \tag{5.2}$$

defined by $(b, c) \rightarrow b^2 c$ relative to the right side of (5.1). We find similarly that evaluation of character functions on $2 \cdot \chi_0$ gives isomorphisms

$$\text{Ad } HCl(\mathbf{Z}) = J(\mathbf{Q})/U(\mathbf{Q})^2 = \mathbf{Q}^* \times \prod_{p < \infty} \mathbf{Z}_p^*/(\mathbf{Z}_p^*)^2. \tag{5.3}$$

The rational classes in $\text{Ad } HCl(\mathbf{Z})$ relative to those description are those of the form $r \times [(\pm 1)_f]$, where $0 < r \in \mathbf{Q}_{>0}$ and $[(\pm 1)_f]$ is the image of $(\pm 1)_f \in \prod_{p < \infty} \mathbf{Z}_p^*$ in the quotient on the far right in (5.3). Suppose L is a finite extension of \mathbf{Q} , and let $t_L : L \times L \rightarrow \mathbf{Q}$ be the trace pairing. The ring of integers O_L of L is a Hermitian module with respect to t_L . In Definition 4.4, we can chose global and local bases to be all equal to a single basis for O_L as a \mathbf{Z} -module. From this definition we now find that the isomorphism in (5.2) identifies the discriminant $d(O_L, t_L)$ with the usual numerical discriminant $d_{L/\mathbf{Q}}$ of L .

6 Epsilon Constants

We will continue to use the notations of §3.

Let V be a virtual representation of G with trivial determinant. Suppose v is a place of \mathbf{Q} , and that l is a prime of \mathbf{Q} which is not associated to v if v is finite. Let dx_v be the standard Haar measure on \mathbf{Z}_v and let ψ_v be the Tate-Iwasawa additive character of \mathbf{Q}_v , as defined in [25, p. 316 - 319], where \mathbf{Z}_v is defined to be \mathbf{Q}_v if v is archimedean. Thus dx_v is self-dual with respect to ψ_v . In §2.4 of [4] we recall the definition of the local epsilon constants $\epsilon_v(X \otimes_G V, \psi_v, dx_v, l)$ and $\epsilon_{v,0}(X \otimes_G V, \psi_v, dx_v, l)$. These definitions involve the virtual representation ${}^l h_{v,l}(V)$ of the Weil-Deligne group ${}^l W_v$ at v which is associated to the V -part of the Euler characteristic of the l -adic cohomology of X (resp. the Hodge realization of the cohomology of X) if v is finite (resp. infinite). If v is a finite place of \mathbf{Q} such that the fiber \mathcal{X}_v of \mathcal{X} at v is smooth, then ${}^l h_{v,l}(V)$ is unramified. If v is infinite, $\epsilon_v(X \otimes_G V, \psi_v, dx_v, l)$ and $\epsilon_{v,0}(X \otimes_G V, \psi_v, dx_v, l)$ do not depend on the choice of l , and this is also conjectured to be the case if v is finite (c.f. [4, Remark 2.3.1(b)]). If v is finite, $\dim(V) = 0$ and ${}^l h_{v,l}(V)$ is unramified, then it follows from [9, §5] that $\epsilon_{v,0}(X \otimes_G V, \psi_v, dx_v, l) = 1$ since we have assumed $\det(V) = 1_G$. We refer the reader to [4, §2 - §3] for further details.

We now return to V of arbitrary dimension (but trivial determinant). If Z is a quasi-projective scheme over $\mathbf{F}_p = \mathbf{Z}/p$ which supports a right action of G , we have an epsilon factor $\epsilon(Z/G, V)$ defined in [6, §2.5]. The function $\epsilon(Z/G)$ defined by $\chi_V \rightarrow \epsilon(Z/G, V^*)$ lies in $\text{Hom}_{\Omega_{\mathbf{Q}}}(R_G, \overline{\mathbf{Q}}^*)$, where V^* is the contragredient of V . Let $\epsilon^s(Z/G)$ be the restriction of this function to R_G^s .

Modifying slightly the notations of [6, Def. 3.1.1], we set

$$\epsilon_v(\mathcal{X} \otimes_G V, \psi_v, dx_v, l) = \epsilon_{v,0}(X \otimes_G V, \psi_v, dx_v, l) \cdot \epsilon(\mathcal{X}_v/G, V) \quad (6.1)$$

if $v = p$ is a finite place of \mathbf{Q} , and $\epsilon_{\infty}(\mathcal{X}, V, \psi_v, dx_v, l) = \epsilon_{\infty,0}(X \otimes_G V, \psi_v, dx_v, l)$ if v is the infinite place ∞ of \mathbf{Q} .

Suppressing the dependence of $\epsilon_{v,0}(X \otimes_G V, \psi_v, dx_v, l)$ on ψ_v, dx_v and l , we will use the notations

$$\epsilon_{v,0}(\mathcal{Y}, V) = \epsilon_{v,0}(X \otimes_G V, \psi_v, dx_v, l) \quad \text{and} \quad \epsilon_v(\mathcal{Y}, V) = \epsilon_v(\mathcal{X} \otimes_G V, \psi_v, dx_v, l). \quad (6.2)$$

As in §8, if f is a function on R_G , we will write \tilde{f} for the function defined by $\tilde{f}(V) = f(V - \dim(V) \cdot 1_G)$. Let S be a finite set of finite places of \mathbf{Q} . Fix a rational prime $l \notin S$ over which \mathcal{X} and \mathcal{Y} are smooth and $\mathcal{X} \rightarrow \mathcal{Y}$ is étale. Let

$$\tilde{\epsilon}_0(\mathcal{Y}, V) = \prod_{v \neq l} \tilde{\epsilon}_{v,0}(\mathcal{Y}, V) \quad \text{and} \quad \tilde{\epsilon}(\mathcal{Y}, V) = \prod_{v \neq l} \tilde{\epsilon}_v(\mathcal{Y}, V) \quad (6.3)$$

and

$$\begin{aligned}
\epsilon_{S,0}(\mathcal{Y}, V) &= \tilde{\epsilon}_0(\mathcal{Y}, V) \cdot \prod_{v \in S} \epsilon_{v,0}(\mathcal{Y}, \dim(V) \cdot 1_G) \\
&= \prod_{v \in S} \epsilon_{v,0}(\mathcal{Y}, V) \cdot \prod_{v \notin S} \tilde{\epsilon}_{v,0}(\mathcal{Y}, V).
\end{aligned} \tag{6.4}$$

Define

$$\epsilon(\mathcal{Y}, V) = \prod_v \epsilon_v(\mathcal{Y}, V). \tag{6.5}$$

Almost all terms in the above products are equal to 1. The values of the products are conjectured to be independent of the choice of l .

By [6, Cor. 1 to Thm. 1], if V is symplectic, then $\tilde{\epsilon}_{v,0}(\mathcal{Y}, V)$ is a non-zero rational number. It is shown in [3, Thm 5.2] that if V is symplectic, and $v = p$ is a finite place of \mathbf{Q} , then $\tilde{\epsilon}(\mathcal{X}_v/G, V)$ is a totally real algebraic number and there is an integer m such that all the finite absolute values of $\tilde{\epsilon}(\mathcal{X}_v/G, V)/p^m$ equal 1. It follows that $\tilde{\epsilon}(\mathcal{X}_v/G, V) = \pm p^m$. By the arguments just after Theorem 7.9 in [7], if p is a rational prime then $\epsilon_{p,0}(\mathcal{Y}, 2 \cdot 1_G, \psi_p, dx_p)$ is ± 1 times an integral power of p . Because $\epsilon(\mathcal{X}_v/G) \in \text{Hom}_{\Omega_{\mathbf{Q}}}(R_G, \overline{\mathbf{Q}}^*)$, the number $\epsilon(\mathcal{X}_v/G)(2 \cdot 1_G) = \epsilon(\mathcal{X}_v/G)(1_G)^2$ is the square of a rational number. Since the dimension of every symplectic representation of G is even, we can therefore make the following definition.

Definition 6.1 *Let $\epsilon_{S,0}^s(\mathcal{Y})$, $\epsilon^s(\mathcal{Y})$ be the elements of $\text{Hom}_{\Omega_{\mathbf{Q}}}(R_G^s, \mathbf{Q}^*)$ defined by the functions $\chi_V \rightarrow \epsilon_{S,0}^s(\mathcal{Y}, V)$ and $\chi_V \rightarrow \epsilon(\mathcal{Y}, V)$, respectively. We will view these character functions as rational classes in the adelic Hermitian classgroup $\text{Ad } HCl(\mathbf{Z}G)$ via the embedding $\Delta_{f,*}$ defined in Proposition 5.1.*

For the remainder of this section, we will make the following assumptions:

Hypothesis 6.2 *The scheme \mathcal{X} is regular and equidimensional of dimension 2. The quotient $\mathcal{Y} = \mathcal{X}/G$ is regular, and for every prime p , the fiber $\mathcal{Y}_p = \mathcal{Y} \otimes_{\mathbf{Z}} \mathbf{F}_p$ of \mathcal{Y} at p has the following properties:*

- a. *The irreducible components of \mathcal{Y}_p have multiplicity prime to p , and*
- b. *The reduction $\mathcal{Y}_p^{\text{red}}$ is a divisor with strictly normal crossings on \mathcal{Y} .*

This implies that the fibers of \mathcal{X} also have properties (i) and (ii), since the action of G on \mathcal{X} has been assumed to be tame.

Let S be the finite set of primes l over which \mathcal{Y} is not smooth or the natural projection $\pi : \mathcal{X} \rightarrow \mathcal{Y}$ is not étale. Define \mathcal{X}_S to be the disjoint union of the fibers of $\mathcal{X}_p = \mathcal{X} \otimes_{\mathbf{Z}} \mathbf{F}_p$ as p ranges over S . Define $\omega_{\mathcal{X},S}^{\bullet}(0)[1]$ to be the complex $[\mathcal{O}_{\mathcal{X}} \xrightarrow{0} \omega_{\mathcal{X}}(\mathcal{X}_S^{\text{red}})]$ having terms in degree -1 and 0 and trivial boundary homomorphism. The restriction to the general fiber X of \mathcal{X} of $\omega_{\mathcal{X},S}^{\bullet}(0)[1]$ is quasi-isomorphic to $[O_X \rightarrow \Omega_{X/\mathbf{Q}}^1]$, which is quasi-isomorphic to the restriction of $\lambda^{\leq d}(\Omega_{\mathcal{X}/\mathbf{Z}}^1)[d]$.

Definition 6.3 Let $R, (\omega_{\mathcal{X},S}^\bullet(0)[1])$ be a perfect complex of $\mathbf{Z}G$ -modules representing the hypercohomology of $\omega_{\mathcal{X},S}^\bullet(0)[1]$. Let \langle, \rangle be the perfect pairing on $H^\bullet(R, (\omega_{\mathcal{X},S}^\bullet(0)[1])_{\mathbf{Q}})$ which results from the pairings of Definition 3.3 and the above quasi-isomorphism between the restrictions to X of $\omega_{\mathcal{X},S}^\bullet(0)[1]$ and $\lambda^{\leq d}(\Omega_{\mathcal{X}/\mathbf{Z}}^1)[d]$. Let

$$\chi_H(\mathcal{X}, G)(0) = d(R, (\omega_{\mathcal{X},S}^\bullet(0)[1]), \langle, \rangle) \quad (6.6)$$

be the associated discriminant in $HCl(\mathbf{Z}G)$; this depends only on \mathcal{X} and the action of G on \mathcal{X} .

Theorem 6.4 Under hypothesis 6.2, the image of $\chi_H(\mathcal{X}, G)(0) \in HCl(\mathbf{Z}G)$ in the adelic Hermitian classgroup $Ad HCl(\mathbf{Z}G)$ is the rational class $\epsilon_{S,0}^s(\mathcal{Y})^{-1}$.

Remark 6.5 For \mathcal{X} of arbitrary dimension, it is shown in [7] that the class $\epsilon_{S,0}^s(\mathcal{Y})^{-1}$ equals a metrized Euler characteristic defined via Arakelov theory. This Euler characteristic is quite different from the one we consider in this paper, however, since it arises from Hermitian metrics on vector bundles and positive definite pairings on cohomology. In dimension 1, the equality of the two kinds of Euler characteristics amounts to the fact that one arrives at the same Hermitian discriminant by considering either the trace form or the Hecke form on rings of integers (c.f. [7, §5.B]).

We now turn to the class $\chi_H(\mathcal{X}, G) \in HCl(\mathbf{Z}G)$ defined in Corollary 3.5. As in [6, §2.5], for each place v of \mathbf{Q} we let $\delta_v : \overline{\mathbf{Q}}^* \rightarrow (\overline{\mathbf{Q}})_v^* = (\overline{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{Q}_v)^*$ and $i_v : (\overline{\mathbf{Q}})_v^* \rightarrow J(\overline{\mathbf{Q}})$ be the natural embeddings. If Z is as above a quasi-projective scheme over \mathbf{F}_p which supports a right action of G , let $\tilde{\epsilon}_v(Z/G)$ be the function in $\text{Hom}(R(G), J(\overline{\mathbf{Q}}))$ defined by $\chi_V \rightarrow i_v \circ \delta_v(\epsilon(Z/G, V^* - \dim(V) \cdot 1))$. As in [6, Def. 2.5.3], $\tilde{\epsilon}_v(Z/G)$ lies in $\text{Hom}_{\Omega_{\mathbf{Q}}}(R(G), J(\overline{\mathbf{Q}}))$.

If $q = r/s$ is a rational number with r and s coprime integers and $s > 0$, set $N(r/s) = r$. For p a rational prime, let $|\tilde{\epsilon}_p(Z/G)|_p(\chi_V)$ be the idele in $J(\overline{\mathbf{Q}})$ whose components at places not over p are 1, and whose component in $(\overline{\mathbf{Q}})_v^* = (\overline{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{Q}_v)^* \subset J(\overline{\mathbf{Q}})$ corresponding to the place w of $\overline{\mathbf{Q}}$ over p is $p^{-\alpha_w}$ where $\alpha_w = N(\text{val}_w(\tilde{\epsilon}(Z/G, V^* - \dim(V) \cdot 1)))$. Here val_w is normalized so that $\text{val}_w(p) = 1$. By [6, Prop. 2.5.1], the resulting character function $|\tilde{\epsilon}_p(Z/G)|_p$ lies in $\text{Hom}_{\Omega_{\mathbf{Q}}}(R(G), J(\overline{\mathbf{Q}}))$. Let $|\tilde{\epsilon}_p^s(Z/G)|_p$ be the restriction of this function to symplectic characters.

For each rational prime p , let $b \subset \mathcal{Y}$ be the reduced branch locus of $\pi : \mathcal{X} \rightarrow \mathcal{Y}$. If Z is the reduction of $\pi^{-1}(b)$, then $b = Z/G$. Let b_p be the fiber of b over \mathbf{F}_p . By assumption, b_p is empty for all but finitely many p .

Definition 6.6 The Hermitian ramification class $HR_{\mathcal{X}}$ is the image in $Ad HCl(\mathbf{Z}G)$ of the character function $\prod_p(\tilde{\epsilon}^s(b_p)|\tilde{\epsilon}_p^s(b_p)|_p)$, where the product is over all rational primes p , and all but finitely many factors in this product are identically 1.

Note that the values of $\tilde{\epsilon}^s(b_p)$ lie in $\overline{\mathbf{Q}}^*$ diagonally embedded into $J(\overline{\mathbf{Q}})$, while the values of $|\tilde{\epsilon}_p^s(b_p)|_p$ are ideles with component 1 outside p . We do not expect in general that $HR_{\mathcal{X}}$ should be a rational class in $Ad HCl(\mathbf{Z}G)$.

Theorem 6.7 Under hypothesis 6.2, the image of $\chi_H(\mathcal{X}, G) \in HCl(\mathbf{Z}G)$ in $Ad HCl(\mathbf{Z}G)$ is $\epsilon^s(\mathcal{Y})^{-1} \cdot HR_{\mathcal{X}}$.

This result parallels the result in $Cl(\mathbf{Z}G)$ proved in [4]. Note, though, that the image of $HR_{\mathcal{X}}$ in $Cl(\mathbf{Z}G)$ will not in general be exactly the ramification class used $R_{\mathcal{X}}$ used in [4].

7 Torsion classes

We will use the notations of Section 3. Thus \mathcal{X} is a flat regular projective scheme over \mathbf{Z} supporting a tame right action by the finite group G . Thus the general fiber $X = \mathcal{X} \otimes_{\mathbf{Z}} \mathbf{Q}$ is smooth, and that G acts freely on X . Let $\mathcal{Y} = \mathcal{X}/G$ and $Y = \mathcal{Y} \otimes_{\mathbf{Z}} \mathbf{Q}$.

Let p be a rational prime, and let $Z = \mathcal{X}_p^{red}$ be the reduced special fiber of \mathcal{X} over p . Suppose T^\bullet is a bounded complex of coherent $\mathcal{O}_{\mathcal{X}}$ - G modules which is supported on \mathcal{X}_p^{red} . The action of G on \mathcal{X}_p^{red} is tame. Let $I = I_p^{red}$ be the ideal sheaf of \mathcal{X}_p^{red} . Then $\{I^j T^\bullet\}_{j=0}^\infty$ is filtration of T^\bullet having finitely many non-zero terms, and each successive quotient $I^j T^\bullet / (I^{j+1} T^\bullet)$ is a bounded complex of coherent \mathcal{O}_Z -modules. By [3, 5], we have for all j a projective Euler characteristic $\chi_p(I^j T^\bullet / I^{j+1} T^\bullet) \in K_0(\mathbf{F}_p G)$. In this way we can associate to T^\bullet the Euler characteristic

$$\chi_p(T^\bullet) = \sum_{j=0}^{\infty} \chi_p(I^j T^\bullet / (I^{j+1} T^\bullet))$$

in $K_0(\mathbf{F}_p G)$. The image of this Euler characteristic in $K_0(\mathbf{Z}G)$ is the usual projective Euler characteristic of T^\bullet .

A result of Nakajima (c.f. [4, Thm. 1.3.2]) shows

Proposition 7.1 (Nakajima) *If the action of G on \mathcal{X}_p^{red} is free, then $\chi_p(T^\bullet)$ is the class of a free $\mathbf{F}_p G$ -module in $K_0(\mathbf{F}_p G)$.*

Let $K_0 T(\mathbf{Z}G)$ be the Grothendieck group of all finite $\mathbf{Z}G$ -modules of finite projective dimension. We have a natural homomorphism $K_0(\mathbf{F}_p G) \rightarrow K_0 T(\mathbf{Z}G)$, and also from [13] a homomorphism $K_0 T(\mathbf{Z}G) \rightarrow HCl(\mathbf{Z}G)$. Let $t_p : K_0(\mathbf{F}_p G) \rightarrow HCl(\mathbf{Z}G)$ be the composition of these two homomorphisms.

Now suppose that L_1^\bullet and L_2^\bullet are bounded complexes of coherent $\mathcal{O}_{\mathcal{X}}$ - G -modules for which there is a distinguished triangle

$$L_1^\bullet \rightarrow L_2^\bullet \rightarrow T^\bullet \tag{7.1}$$

in the derived category $D(\mathcal{X}, G)$. The resulting distinguished triangle in cohomology

$$R, (L_1^\bullet) \rightarrow R, (L_2^\bullet) \rightarrow R, (T^\bullet) \tag{7.2}$$

induces an isomorphism $R, (L_1^\bullet)_{\mathbf{Q}} \rightarrow R, (L_2^\bullet)_{\mathbf{Q}}$ in the derived category $D(\mathbf{Q}G)$. This isomorphism allows us to identify perfect pairings \langle, \rangle (in the sense of Definition 2.6) on $R, (L_1^\bullet)_{\mathbf{Q}}$ and $R, (L_2^\bullet)_{\mathbf{Q}}$.

Proposition 7.2 *With the above notation we have an equality of discriminants*

$$d(L_2^\bullet, \langle, \rangle) = d(L_1^\bullet, \langle, \rangle) \cdot t_p(\chi_p(T^\bullet)). \tag{7.3}$$

The same equality holds if one has a distinguished triangle

$$T^\bullet \rightarrow L_2^\bullet \rightarrow L_1^\bullet \tag{7.4}$$

instead of (7.1).

Proof: By induction, it is enough to consider the case in which T^\bullet is a complex of coherent \mathcal{O}_Z - G -modules. We can find bounded complexes P^\bullet (resp. R^\bullet) of finitely generated projective $\mathbf{Z}G$ -modules (resp. $\mathbf{F}_p G$ -modules) representing R , (L_2^\bullet) (resp. R , (T^\bullet)). The projectivity of the terms of P^\bullet implies that we can find a morphism of complexes $\tau : P^\bullet \rightarrow R^\bullet$ representing the morphism in the derived category appearing in (7.2). Adding a suitable bounded acyclic complex of finitely generated free modules to P^\bullet , we can adjust τ so that it gives surjections $\tau_i : P^i \rightarrow R^i$ between the terms of P^\bullet and R^\bullet . The complex C^\bullet formed by the kernels of the τ_i now gives a distinguished triangle $C^\bullet \rightarrow P^\bullet \rightarrow R^\bullet$ in $D(\mathbf{Z}G)$, and this triangle is isomorphic to (7.2). The terms of C^\bullet are projective $\mathbf{Z}G$ -modules since

$$0 \rightarrow C^i \rightarrow P^i \rightarrow R^i \rightarrow 0 \quad (7.5)$$

is exact for all i . Since $R_{\mathbf{Q}}^\bullet$ is the zero complex, $C_{\mathbf{Q}}^\bullet \rightarrow P_{\mathbf{Q}}^\bullet$ is an isomorphism. Using the procedure in Proposition 2.7, we can now simultaneously adjust C^\bullet and P^\bullet so that there is a perfect morphism $\phi : P_{\mathbf{Q}}^\bullet \rightarrow \widetilde{P}_{\mathbf{Q}}^\bullet$ giving rise to the perfect pairing $\langle \cdot, \cdot \rangle$. We now use the isomorphism $C_{\mathbf{Q}}^\bullet = P_{\mathbf{Q}}^\bullet$ to compute the discriminants $d(C^\bullet, \langle \cdot, \cdot \rangle) = d(C^\bullet, \phi)$ and $d(P^\bullet, \langle \cdot, \cdot \rangle) = d(P^\bullet, \phi)$. Global $\mathbf{Q}G$ -bases for the terms of $C_{\mathbf{Q}}^\bullet$ give such bases for the terms of $P_{\mathbf{Q}}^\bullet$. Hence the recipe for computing Hermitian discriminants via choices of global and local bases, together with the exact sequences (7.5) show (7.3), since $\chi_p(T^\bullet) = \sum_i (-1)^i [R_i]$. If one assumes (7.1) rather than (7.4), one proceeds similarly, using the distinguished triangle

$$R, (L_2^\bullet) \rightarrow R, (L_1^\bullet) \rightarrow R, (T^\bullet)[-1] \quad (7.6)$$

instead of (7.2). In this case one takes P^\bullet to represent R , (L_1^\bullet) , and uses the morphism $P^\bullet \rightarrow R^\bullet[-1]$ associated to (7.6).

8 Hermitian classes of degree 0.

Let F be a number field. We will use the following conventions from [7, §4.E] concerning functions on the character group R_G of a group G . If A is an abelian group and $f \in \text{Hom}(R_G, A)$, write $\tilde{f} \in \text{Hom}(R_G, A)$ for the homomorphism defined by $\tilde{f}(\chi) = f(\chi - \chi(1) \cdot 1)$. We will use the same convention for elements f of $\text{Hom}(R_G^s, A)$. As noted in [7, §4.E], the map $f \rightarrow \tilde{f}$ induced a well defined endomorphism $c \rightarrow \tilde{c}$ of $HCl(O_F G)$ and $Ad HCl(O_F G)$. Clearly $c \rightarrow \tilde{c}$ sends a rational class in $Ad HCl(\mathbf{Z}G)$ to a rational class (c.f. §5).

We have a homomorphisms

$$\text{Ind} : HCl(O_F) \rightarrow HCl(O_F G) \quad \text{and} \quad \text{Coinf} : HCl(O_F G) \rightarrow HCl(O_F)$$

given in terms of character functions by

$$\text{Ind}(f)(\psi) = f(\text{Res}_G^e \psi) = f(\psi(e) \cdot 1_{\{e\}}) \quad \text{and} \quad \text{Coinf}(f)(\psi') = f(\text{Infl}_{\{e\}}^G \psi')$$

for $\psi \in R_G$ and $\psi' \in R_{\{e\}}$. Here 1_G denotes the trivial character of G , $\psi' = \psi'(e) \cdot 1_{\{e\}}$ and $\text{Infl}_{\{e\}}^G \psi' = \psi'(e) \cdot 1_G$.

Suppose (P^\bullet, ϕ) is a perfect Hermitian complex of $\mathbf{Z}G$ -modules, and that $\langle \cdot, \cdot \rangle = \{\langle \cdot, \cdot \rangle_t\}_t$ is the perfect pairing on the cohomology $H^\bullet(P_\mathbf{Q}^\bullet)$ corresponding to ϕ . Let $(P^\bullet)^G$ (resp. $(P^\bullet)_G$) be the complex of \mathbf{Z} -modules formed by the G -invariants (resp. G -covariants) of the terms of P^\bullet . The natural morphisms $(P^\bullet)^G \rightarrow P^\bullet$ and $P^\bullet \rightarrow (P^\bullet)_G$ induce identifications $H^t((P^\bullet)^G)_\mathbf{Q} = H^t((P^\bullet)_\mathbf{Q})^G$ and $H^t((P^\bullet)_\mathbf{Q})_G = H^t(((P^\bullet)_G)_\mathbf{Q})$.

The morphism $\phi : P_\mathbf{Q}^\bullet \rightarrow \tilde{P}_\mathbf{Q}^\bullet = \text{Hom}_\mathbf{Q}(P_\mathbf{Q}^\bullet, \mathbf{Q})$ induces a morphism of complexes $\phi^G : (P_\mathbf{Q}^\bullet)^G \rightarrow \text{Hom}_\mathbf{Q}((P_\mathbf{Q}^\bullet)^G, \mathbf{Q}) = \text{Hom}_\mathbf{Q}(((P^\bullet)_G)_\mathbf{Q}, \mathbf{Q})$. The norm map in the group ring $\mathbf{Z}G$ gives an isomorphism $((P^\bullet)_G)_\mathbf{Q} \rightarrow ((P^\bullet)^G)_\mathbf{Q}$. Using this norm map, we will view ϕ^G as a morphism

$$\phi^G : ((P^\bullet)^G)_\mathbf{Q} \rightarrow ((P^\bullet)^G)_\mathbf{Q} = \text{Hom}_\mathbf{Q}(((P^\bullet)^G)_\mathbf{Q}, \mathbf{Q})$$

of complexes of vector spaces over \mathbf{Q} . Since (P^\bullet, ϕ) is a perfect Hermitian complex of $\mathbf{Z}G$ -modules, $((P^\bullet)^G, \phi^G)$ is a perfect Hermitian complex of \mathbf{Z} -modules. Let

$$\langle \cdot, \cdot \rangle_t^G : H^t((P^\bullet)^G)_\mathbf{Q} \times H^t((P^\bullet)^G)_\mathbf{Q} \rightarrow \mathbf{Q}$$

be the pairing induced by ϕ^G . Then $\langle \cdot, \cdot \rangle^G = \{\langle \cdot, \cdot \rangle_t^G\}_t$ is a perfect pairing for the action of the trivial group $\{e\}$ on the cohomology $H^\bullet((P^\bullet)^G)_\mathbf{Q}$. We have a Hermitian discriminant $d((P^\bullet)^G, \phi^G) = d((P^\bullet)^G, \langle \cdot, \cdot \rangle^G)$ in the Hermitian class group $HCl(\mathbf{Z})$.

Suppose $m \in H^t(((P^\bullet)^G)_\mathbf{Q}) = H^t(P_\mathbf{Q}^\bullet)^G$ and $m' \in H^{-t}(((P^\bullet)^G)_\mathbf{Q}) = H^{-t}(P_\mathbf{Q}^\bullet)^G$. Since the norm in the group ring of $\mathbf{Q}G$ acts by multiplication by $\#G$ on $H^{-t}(P_\mathbf{Q}^\bullet)^G$, we find via the above identifications that

$$\#G \cdot \langle m, m' \rangle_t^G = \langle m, m' \rangle_t . \quad (8.1)$$

Proposition 8.1 *Let $c = d(P^\bullet, \langle \cdot, \cdot \rangle)$ and $c_0 = d((P^\bullet)^G, \langle \cdot, \cdot \rangle^G)$. Then*

$$\text{Coinf}(c) = c_0 \quad \text{and} \quad c = \tilde{c} \cdot \text{Ind}(c_0).$$

Proof: We consider the case in which P^\bullet has a single non-zero term $M = P_0$ in degree 0, the general case being similar. Let $F = \mathbf{Q}$ in Definition 4.4, and let $h = h_\mathbf{Q} : M_\mathbf{Q} \times M_\mathbf{Q} \rightarrow \mathbf{Q}G$ be the Hermitian pairing resulting from $\langle \cdot, \cdot \rangle_0$. Thus

$$h(m, m') = \sum_{g \in G} \langle m, gm' \rangle_0 g^{-1}$$

for $m, m' \in M_\mathbf{Q}$. Let $h^G : M_\mathbf{Q}^G \times M_\mathbf{Q}^G \rightarrow \mathbf{Q}$ be the pairing resulting from $\langle \cdot, \cdot \rangle_0^G$. Then for $m, m' \in M_\mathbf{Q}^G$, we find from (8.1) that

$$h^G(m, m')e_G = \langle m, m' \rangle_0^G e_G = \frac{1}{\#G} \sum_{g \in G} \langle m, gm' \rangle_0^G g^{-1} = \frac{1}{(\#G)^2} h(m, m') \quad (8.2)$$

where e_G is the idempotent of the trivial representation of G .

As in Definition 4.4, let $\{v_i\}_{i=1}^q$ be a basis for $M_\mathbf{Q}$ as a free $\mathbf{Q}G$ -module, and let $\lambda = (\lambda_v)_v$ be an element of $GL_q(\text{Ad}(\mathbf{Q}) \otimes_\mathbf{Q} \mathbf{Q}G)$ such that λ_v takes $\{v_i\}_{i=1}^q$ to a basis for $\mathbf{Z}_v \otimes_\mathbf{Z} M$ as a free $\mathbf{Z}_v G$ -module for each finite place v of \mathbf{Q} . Then $c = d(M, h)$ in $HCl(\mathbf{Z}G)$ is represented by the class

$$(\text{Det}(\lambda) \ , \ Pf((h(v_i, v_j))_{i,j}) \in \frac{\text{Det}(J(\mathbf{Q}G))}{\text{Det}(U(\mathbf{Z}G))} \times \text{Hom}_{\Omega_\mathbf{Q}}(R_G^s, \overline{\mathbf{Q}}^*). \quad (8.3)$$

We can choose $\{\#Ge_G v_i\}_{i=1}^q$ as our basis for $M_{\mathbf{Q}}^G$ as free \mathbf{Q} -module. Since

$$\mathbf{Z}_v \otimes_{\mathbf{Z}} M^G = \#Ge_G(\mathbf{Z}_v \otimes_{\mathbf{Z}} M)$$

we find that $e_G \lambda \in GL_q(Ad(\mathbf{Q}))$ takes $\{\#Ge_G v_i\}_{i=1}^q$ to local bases for $\mathbf{Z}_v \otimes_{\mathbf{Z}} M^G$ at all finite places v of \mathbf{Q} . Thus c_0 is represented by the class

$$(\text{Det}(e_G \lambda) , Pf((h^G(\#Gv_i, \#Gv_j))_{i,j})) \in \frac{\text{Det}(J(\mathbf{Q}))}{\text{Det}(U(\mathbf{Q}))} \times \text{Hom}(R_{\{e\}}^s, \overline{\mathbf{Q}}^*). \quad (8.4)$$

Viewing $e_G \lambda$, as above, as an element of $GL_q(Ad(\mathbf{Q}))$, we have

$$\text{Det}(\lambda)(1_G) = \text{Det}(e_G \lambda)(1_{\{e\}}). \quad (8.5)$$

Since the group of symplectic characters of the trivial group is generated by $2 \cdot 1_{\{e\}}$, we see from (8.3),(8.4) and (8.5) that Proposition 8.1 will hold if we can show

$$Pf((h(v_i, v_j))_{i,j})(2 \cdot 1_G) = Pf((h^G(\#Ge_G v_i, \#Ge_G v_j))_{i,j})(2 \cdot 1_{\{e\}}). \quad (8.6)$$

In view of [13, Prop. 4.6], this equality is equivalent to

$$\det((c_{i,j})_{i,j}) = \det((h^G(\#Ge_G v_i, \#Ge_G v_j))_{i,j}) \quad (8.7)$$

where $c_{i,j} \in \mathbf{Q}$ and $c_{i,j} e_G = e_G h(v_i, v_j)$. Formula (8.2) shows

$$h^G(\#Ge_G v_i, \#Ge_G v_j) e_G = \frac{1}{(\#G)^2} h(\#Ge_G v_i, \#Ge_G v_j) = h(v_i, v_j) e_G$$

where the last equality follows from the fact that h is a Hermitian form. This implies (8.7), so Proposition 8.1 is proved.

9 Reduction to degree 0.

For the rest of this paper we will assume Hypothesis 6.2, and we let S be a finite set of rational primes chosen as in §6. Our goal in this section is to prove:

Proposition 9.1 *The following statements are equivalent to Theorem 6.4 and Theorem 6.7, respectively:*

- a. *The image of the degree 0 class $\tilde{\chi}_H(\mathcal{X}, G)(0) \in HCl(\mathbf{Z}G)$ in the adelic Hermitian classgroup $Ad HCl(\mathbf{Z}G)$ is the rational class $\tilde{\epsilon}_{S,0}^s(\mathcal{Y})^{-1}$.*
- b. *The image of the degree 0 class $\tilde{\chi}_H(\mathcal{X}, G) \in HCl(\mathbf{Z}G)$ in $Ad HCl(\mathbf{Z}G)$ is $\tilde{\epsilon}^s(\mathcal{Y})^{-1} \cdot HR_{\mathcal{X}}$.*

Lemma 9.2 *Let $\pi_{\mathbf{Q}} : X \rightarrow Y$ be the morphism on generic fibers induced by $\pi : \mathcal{X} \rightarrow \mathcal{Y}$. We have an isomorphism of sheaves $\pi_{\mathbf{Q}}^* \Omega_{Y/\mathbf{Q}}^1 \rightarrow \Omega_{X/\mathbf{Q}}^1$ which results from the inclusion $\pi_{\mathbf{Q}}$ induces of the function ring of Y into that of X . This isomorphism induces an isomorphism between $\pi^* \omega_{\mathcal{Y}}(\mathcal{Y}_S^{red})$ and $\omega_{\mathcal{X}}(\mathcal{X}_S^{red})$.*

Proof: By [7, Lemma 7.10], the sheaf of relative degree one logarithmic differentials $\Omega_{\mathcal{X}/\mathbf{Z}}^1(\log \mathcal{X}_S^{red}/\log S)$ is canonically isomorphic to $\omega_{\mathcal{X}}(\mathcal{X}_S^{red} - \mathcal{X}_S)$. Since $\pi : \mathcal{X} \rightarrow \mathcal{Y}$ is log-étale, we find

$$\pi^* \Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{red}/\log S) = \Omega_{\mathcal{X}/\mathbf{Z}}^1(\log \mathcal{X}_S^{red}/\log S).$$

Since $\pi^* \mathcal{O}_{\mathcal{Y}}(\mathcal{Y}_S) = \mathcal{O}_{\mathcal{X}}(\mathcal{X}_S)$, this implies $\pi^* \omega_{\mathcal{Y}}(\mathcal{Y}_S^{red}) = \omega_{\mathcal{X}}(\mathcal{X}_S^{red})$.

Lemma 9.3 *The natural morphism from the complex $\omega_{\mathcal{Y},S}^\bullet(0)[1] = [\mathcal{O}_{\mathcal{Y}} \rightarrow \omega_{\mathcal{Y}}(\mathcal{Y}_S^{red})]$ of $\mathcal{O}_{\mathcal{Y}}$ -modules to the complex $\omega_{\mathcal{X},S}^\bullet(0)[1]^G = [\mathcal{O}_{\mathcal{X}}^G \rightarrow \omega_{\mathcal{X}}(\mathcal{X}_S^{red})^G]$ formed by the G -invariants of the terms of $\omega_{\mathcal{X},S}^\bullet(0)[1]$ is an isomorphism.*

Proof: This is clear from the fact that \mathcal{X} is flat over \mathcal{Y} and $\omega_{\mathcal{X},S}^\bullet(0)[1] = \mathcal{O}_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{Y}}} \omega_{\mathcal{Y},S}^\bullet(0)[1]$.

Lemma 9.4 *Let $R, (\omega_{\mathcal{X},S}^\bullet(0)[1])$ (resp. $R, (\omega_{\mathcal{Y},S}^\bullet(0)[1])$) be a perfect complex of $\mathbf{Z}G$ -modules (resp. \mathbf{Z} -modules) representing the hypercohomology of $\omega_{\mathcal{X},S}^\bullet(0)[1]$ (resp. $\omega_{\mathcal{Y},S}^\bullet(0)[1]$). Let \langle, \rangle be the perfect pairing on $R, (\omega_{\mathcal{X},S}^\bullet(0)[1])_{\mathbf{Q}}$ specified in Definition 6.3, and let \langle, \rangle_Y be the corresponding pairing on $R, (\omega_{\mathcal{Y},S}^\bullet(0)[1])_{\mathbf{Q}}$ for the trivial group action on \mathcal{Y} . There is a quasi-isomorphism between $(R, (\omega_{\mathcal{X},S}^\bullet(0)[1]))^G$ and $R, (\omega_{\mathcal{Y},S}^\bullet(0)[1])$ which identifies perfect pairing \langle, \rangle^G defined by (8.1) with the pairing \langle, \rangle_Y .*

Proof By computing cohomology via a G -equivariant Čech complex, one sees that $(R, (\omega_{\mathcal{X},S}^\bullet(0)[1]))^G$ is quasi-isomorphic to $R, (\omega_{\mathcal{X},S}^\bullet(0)[1]^G)$, and thus to $R, (\omega_{\mathcal{Y},S}^\bullet(0)[1])$ by Lemma 9.3 (compare [7, Lemma 7.6]). We must now show the following for all integers t : suppose $m \in H^t(\mathcal{Y}, \omega_{\mathcal{Y},S}^\bullet(0)[1]) \otimes_{\mathbf{Z}} \mathbf{Q}$ and $m' \in H^{-t}(\mathcal{Y}, \omega_{\mathcal{Y},S}^\bullet(0)[1]) \otimes_{\mathbf{Z}} \mathbf{Q}$. Let $\langle, \rangle_{t,Y}$ be the duality pairing for \mathcal{Y} (with trivial group action) defined just after Hypothesis 6.2. View m and m' as elements of $H^{\pm t}(\mathcal{X}, \omega_{\mathcal{X},S}^\bullet(0)[1]) \otimes_{\mathbf{Z}} \mathbf{Q}$ via the isomorphism $H^{\pm t}(\mathcal{Y}, \omega_{\mathcal{Y},S}^\bullet(0)[1]) \otimes_{\mathbf{Z}} \mathbf{Q} = (H^{\pm t}(\mathcal{X}, \omega_{\mathcal{X},S}^\bullet(0)[1]) \otimes_{\mathbf{Z}} \mathbf{Q})^G$ induced by the above quasi-isomorphism between $R, (\omega_{\mathcal{X},S}^\bullet(0)[1])^G$ and $R, (\omega_{\mathcal{Y},S}^\bullet(0)[1])$. Then by (8.1), it will suffice to show

$$\#G \cdot \langle m, m' \rangle_{t,Y} = \langle m, m' \rangle_t. \quad (9.1)$$

This property follows by computing each pairing via residues (c.f. see [18, p. 247 - 248]) and using the fact that $X \rightarrow Y$ is an étale G -cover. The factor of $\#G$ on the left in (9.1) results from the fact that the residues of differentials of Y at points of X over a given point of Y are equal. For more details on the computation of the pairings via residues, see §11.

Lemma 9.5 *Let $\omega_{\mathcal{Y}}^\bullet[1]$ be the complex $[\mathcal{O}_{\mathcal{Y}} \rightarrow \omega_{\mathcal{Y}}]$ having terms in degrees -1 and 0 . Give $R, (\omega_{\mathcal{Y}}^\bullet[1])_{\mathbf{Q}}$ the perfect pairing \langle, \rangle'_Y resulting from \langle, \rangle_Y and from the natural isomorphism between the restrictions to the general fiber Y of \mathcal{Y} of $\omega_{\mathcal{Y}}^\bullet[1]$ and $\omega_{\mathcal{Y},S}^\bullet(0)[1]$. Then in $HCl(\mathbf{Z}) = \mathbf{Q}^*$ (c.f. Example 5.2) we have $d(\omega_{\mathcal{Y}}^\bullet[1], \langle, \rangle'_Y) = 1$.*

Proof: The cohomology group $H^j(\omega_{\mathcal{Y}}^\bullet[1])$ is naturally isomorphic to $H^0(\mathcal{O}_{\mathcal{Y}})$ if $j = -1$, to

$$H^1(\mathcal{O}_{\mathcal{Y}}) \oplus H^0(\omega_{\mathcal{Y}})$$

if $j = 0$, to $H^1(\omega_{\mathcal{Y}})$ if $j = 1$ and to 0 for all other j . By duality (c.f. [1, p. 302]) the natural pairings

$$H^i(\mathcal{O}_{\mathcal{Y}}) \times H^{1-i}(\omega_{\mathcal{Y}}) \rightarrow H^1(\omega_{\mathcal{Y}}) = \mathbf{Z}$$

are perfect pairings of finitely generated free abelian groups, where $H^1(\omega_{\mathcal{Y}}) = \mathbf{Z}$ is the trace isomorphism. It follows that we can take $R, (\mathcal{Y}, \omega_{\mathcal{Y}}^{\bullet}[1])$ to be represented by the perfect complex $[H^0(\mathcal{O}_{\mathcal{Y}}) \rightarrow H^1(\mathcal{O}_{\mathcal{Y}}) \oplus H^0(\omega_{\mathcal{Y}}) \rightarrow H^1(\omega_{\mathcal{Y}})](1)$ of free \mathbf{Z} -modules, with the pairing $\langle, \rangle_{\mathcal{Y}}$ as above. Thus $d(\omega_{\mathcal{Y}}^{\bullet}[1], \langle, \rangle)$ is a product of discriminants of hyperbolic free \mathbf{Z} -modules (c.f. [13, p. 52]), and such discriminants are trivial by [13, Prop. II.5.5].

Proof of Proposition 9.1:

To prove part (i) of Proposition 9.1, it will suffice by Proposition 8.1 to show $\chi_H(\mathcal{X}, G)(0)$ and $\epsilon_{S,0}^s(\mathcal{Y})^{-1}$ have the same image under the homomorphism $\text{Coinf} : HCl(\mathbf{Z}G) \rightarrow HCl(\mathbf{Z})$ which results from inflating characters from the trivial quotient group of G to G . By Proposition 8.1 and Lemma 9.4, $\text{Coinf}(\chi_H(\mathcal{X}, G)(0))$ is the class $\chi_H(\mathcal{Y}, \{e\})(0)$ in $HCl(\mathbf{Z})$, where $\{e\}$ is the trivial group acting on \mathcal{Y} . This class is the discriminant $d(\omega_{\mathcal{Y},S}^{\bullet}(0)[1], \langle, \rangle_Y)$, where \langle, \rangle_Y is induced by duality on the general fiber of \mathcal{Y} . Thus we need to show

$$d(\omega_{\mathcal{Y},S}^{\bullet}(0)[1], \langle, \rangle_Y) = \text{Coinf}(\epsilon_{S,0}^s(\mathcal{Y})^{-1}) \quad (9.2)$$

in $Ad HCl(\mathbf{Z})$.

By the adjunction formula, we have an exact sequence of sheaves on \mathcal{Y}

$$0 \rightarrow \omega_{\mathcal{Y}} \rightarrow \omega_{\mathcal{Y}}(\mathcal{Y}_S^{red}) \rightarrow \omega_{\mathcal{Y}_S^{red}} \rightarrow 0$$

in which $\omega_{\mathcal{Y}_S^{red}}$ is the relative dualizing sheaf of \mathcal{Y}_S^{red} over \mathbf{Z} . The complexes $\omega_{\mathcal{Y},S}^{\bullet}(0)[1]$ and $\omega_{\mathcal{Y}}^{\bullet}[1]$ differ by having $\omega_{\mathcal{Y}}(\mathcal{Y}_S^{red})$ and $\omega_{\mathcal{Y}}$ in degree 0, respectively. Since $\omega_{\mathcal{Y}_S^{red}}$ is supported on \mathcal{Y}_S , we find from Proposition 7.2 that

$$d(\omega_{\mathcal{Y},S}^{\bullet}(0)[1], \langle, \rangle) = d(\omega_{\mathcal{Y}}^{\bullet}[1], \langle, \rangle) \cdot \prod_{p \in S} t_p(\chi_p(\omega_{\mathcal{Y}_p^{red}})) \quad (9.3)$$

where $\chi_p(\omega_{\mathcal{Y}_p^{red}}) \in K_0(\mathbf{F}_p)$ is the Euler characteristic relative to \mathbf{F}_p of $\omega_{\mathcal{Y}_p^{red}}$ and $t_p : K_0(\mathbf{F}_p) \rightarrow HCl(\mathbf{Z})$ is the homomorphism described in §7. Since \mathcal{Y}_p^{red} is a reduced fibral divisor on \mathcal{Y} with normal crossings, we have $\chi_p(\omega_{\mathcal{Y}_p^{red}}) = -\chi_p(\mathcal{O}_{\mathcal{Y}_p^{red}})$ by duality. This, together with Lemma 9.5 and (9.3), gives

$$d(\omega_{\mathcal{Y},S}^{\bullet}(0)[1], \langle, \rangle) = \prod_{p \in S} t_p(\chi_p(\mathcal{O}_{\mathcal{Y}_p^{red}}))^{-1}. \quad (9.4)$$

The class $\text{Coinf}(\epsilon_{S,0}^s(\mathcal{Y}))$ is in the group $\text{Hom}(R_{\{e\}}^s, \mathbf{Q}^*)$ of rational classes in $Ad HCl(\mathbf{Z})$, where the group $R_{\{e\}}^s$ of symplectic characters of the trivial group $\{e\}$ is generated by $2 \cdot 1_{\{e\}}$ when $1_{\{e\}}$ is the trivial character. By [7, Thm. 7.9], $\epsilon_{S,0}^s(\mathcal{Y}, 2 \cdot 1_{\{e\}})$ is a positive rational number, and we have

$$\text{Coinf}(\epsilon_{S,0}^s(\mathcal{Y})^{-1})(2 \cdot 1_{\{e\}}) = \prod_{p \in S} p^{\chi(Y) + \sum_{i \in I_p} (1-m_i)\chi_c(\mathcal{Y}_{p,i}^*)} \quad (9.5)$$

where $\{\mathcal{Y}_{p,i}\}_{i \in I_p}$ is the set of irreducible components of the special fiber \mathcal{Y}_p , m_i is the multiplicity of $\mathcal{Y}_{p,i}$ in \mathcal{Y}_p , $\mathcal{Y}_{p,i}^*$ is the complement in $\mathcal{Y}_{p,i}$ of the intersection of $\mathcal{Y}_{p,i}$ with other irreducible components of \mathcal{Y}_p , $\chi_c(\mathcal{Y}_{p,i}^*)$ is the étale Euler characteristic of $\mathcal{Y}_{p,i}^*$ with compact supports, and $\chi(Y)$ is the Euler characteristic of the general fiber $Y = \mathcal{Y} \otimes_{\mathbf{Z}} \mathbf{Q}$. We have assumed that the special fibers of \mathcal{Y} are divisors on \mathcal{Y} with strict normal crossing and multiplicities prime to p . Hence there the Swan conductor associated to \mathcal{Y} is trivial, and we have from [22, Cor. 2, p. 407] that for each p ,

$$\chi(Y) = \sum_{i \in I_p} m_i \chi_c(\mathcal{Y}_{p,i}^*). \quad (9.6)$$

Combining (9.5) and (9.6) gives

$$\text{Coinf}(\epsilon_{S,0}^s(\mathcal{Y})^{-1})(2 \cdot 1_{\{e\}}) = \prod_{p \in S} p^{\sum_{i \in I_p} \chi_c(\mathcal{Y}_{p,i}^*)}. \quad (9.7)$$

We claim that

$$2 \cdot \chi_p(\mathcal{O}_{\mathcal{Y}_p^{red}}) = \sum_{i \in I_p} \chi_c(\mathcal{Y}_{p,i}^*) \quad (9.8)$$

when we identify $K_0(\mathbf{F}_p)$ with \mathbf{Z} by sending the class of \mathbf{F}_p to 1. Recall that \mathcal{Y}_p^{red} was assumed to be a divisor with strict normal crossings on \mathcal{Y} . Let \mathbf{F}_{q_i} be field of constants of $\mathcal{Y}_{p,i}$. Let g_i be the genus of $\mathcal{Y}_{p,i}$. Then

$$\chi_c(\mathcal{Y}_{p,i}^*) = (2 - 2g_i)[\mathbf{F}_{q_i} : \mathbf{F}_p] - \sum_{j \neq i} \sum_{x \in \mathcal{Y}_{p,j} \cap \mathcal{Y}_{p,i}} [k(x) : \mathbf{F}_p]. \quad (9.9)$$

The normalization $\mathcal{Y}_p^{red,\#}$ of \mathcal{Y}_p^{red} is the disjoint union of the smooth projective curves $\{\mathcal{Y}_{p,i}\}_i$. The quotient of the direct image on \mathcal{Y}_p^{red} of the structure sheaf of $\mathcal{Y}_p^{red,\#}$ by the subsheaf formed by the structure sheaf of \mathcal{Y}_p^{red} is the direct sum over the singular points of \mathcal{Y}_p^{red} of the direct images of the structure sheaves of these singular points. On taking Euler characteristics, we find

$$\sum_{i \in I_p} (1 - g_i)[\mathbf{F}_{q_i} : \mathbf{F}_p] = \chi_p(\mathcal{O}_{\mathcal{Y}_p^{red,\#}}) = \chi_p(\mathcal{O}_{\mathcal{Y}_p^{red}}) + s \quad (9.10)$$

where s is the sum of the degrees over \mathbf{F}_p of the singular points of \mathcal{Y}_p^{red} . Since

$$2s = \sum_{j \neq i \in I_p} \sum_{x \in \mathcal{Y}_{p,j} \cap \mathcal{Y}_{p,i}} [k(x) : \mathbf{F}_p] \quad (9.11)$$

we deduce (9.8) from (9.9), (9.10) and (9.11).

Our normalization of the homomorphism $t_p : K_0(\mathbf{F}_p) \rightarrow HCl(\mathbf{Z})$ implies that $t_p(\mathbf{F}_p)$ is the difference of the Hermitian classes (\mathbf{Z}, h_0) and $(p\mathbf{Z}, h_0)$, where h_0 is the Hermitian pairing on \mathbf{Z} defined by $h(1, 1) = 1$. One finds that in $Ad HCl(\mathbf{Z})$, $t_p(\mathbf{F}_p)$ is the rational class in $\text{Hom}(R_{\{e\}}^s, \mathbf{Q}^*)$ which sends $2 \cdot 1_{\{e\}}$ to p^{-2} . Combining this with (9.8) and (9.7) shows

$$\text{Coinf}(\epsilon_0^s(\mathcal{Y})^{-1}) = \prod_{p \in S} t_p(\chi_p(\mathcal{O}_{\mathcal{Y}_p^{red}}))^{-1} \quad (9.12)$$

in $HCl(\mathbf{Z})$. Combining (9.12) and (9.4) shows the equality (9.2), which completes the proof of part (i) of Proposition 9.1.

Part (ii) of Proposition 9.1 is proved in a similar way, so we will only sketch this. By Proposition 9.1 it will suffice to show

$$\text{Coinf}(\chi_H(\mathcal{X}, G)) = \text{Coinf}(\epsilon^s(\mathcal{Y})^{-1} \cdot HR_{\mathcal{X}}) = \text{Coinf}(\epsilon^s(\mathcal{Y})^{-1}) \quad (9.13)$$

in $HCl(\mathbf{Z}G)$, where the second equality holds by because the character function defining $HR_{\mathcal{X}}$ in Definition 6.6 is the identity on the trivial character. The calculations of the proof of [7, Thm. 8.5] show that the natural homomorphism of coherent sheaves $\Omega_{\mathcal{Y}/\mathbf{Z}}^1 \rightarrow \pi_*(\Omega_{\mathcal{X}/\mathbf{Z}}^1)^G$ on \mathcal{Y} is an isomorphism. It follows, on considering duality pairings, that

$$\text{Coinf}(\chi_H(\mathcal{X}, G)) = \chi_H(\mathcal{Y}, \{e\})$$

where $\{e\}$ is the trivial group acting on \mathcal{Y} . In this way, we can reduce to proving (9.13) when $\mathcal{X} = \mathcal{Y}$ and G is the trivial group.

The natural homomorphism $\Omega_{\mathcal{Y}/\mathbf{Z}}^1 \rightarrow \omega_{\mathcal{Y}/\mathbf{Z}}$ is an isomorphism on the general fiber of \mathcal{Y} . Let $\mathcal{C}_{\mathcal{Y}}$ be the complex in degrees -1 and 0 defined by this homomorphism. Then $\mathcal{C}_{\mathcal{Y}}$ has an Euler characteristic in the Grothendieck group $K_0T(\mathbf{Z})$ of all finite \mathbf{Z} -modules. By [1, Thm. 2.3], this Euler characteristic is equal to the class of $\mathbf{Z}/A(\mathcal{Y})\mathbf{Z}$, where $A(\mathcal{Y}) > 0$ is the conductor of \mathcal{Y} . The complex $\omega_{\mathcal{Y}}^{\bullet}[1]$ has $\mathcal{O}_{\mathcal{Y}}$ in degree -1 and $\omega_{\mathcal{Y}}$ in degree 0 . It follows from Proposition 7.2 that in $Ad HCl(\mathbf{Z})$ we have

$$d(\omega_{\mathcal{Y}}^{\bullet}[1], \langle, \rangle) / \chi_H(\mathcal{Y}, \{e\}) = t(\mathbf{Z}/A(\mathcal{Y})\mathbf{Z}) \quad (9.14)$$

where $t : K_0T(\mathbf{Z}) \rightarrow Ad HCl(\mathbf{Z})$ is the homomorphism defined in §7. We know by Lemma 9.5 that $d(\omega_{\mathcal{Y}}^{\bullet}[1], \langle, \rangle) = 1$, while $t(\mathbf{Z}/A(\mathcal{Y})\mathbf{Z})$ is the rational class in $Ad HCl(\mathbf{Z})$ defined by the character function sending $2 \cdot 1_{\{e\}}$ to $A(\mathcal{Y})^{-2}$. These facts and (9.14) reduce the proof of (9.13) to showing

$$\epsilon(\mathcal{Y}, 1_{\{e\}})^{-2} = \text{Coinf}(\epsilon^s(\mathcal{Y})^{-1})(2 \cdot 1_{\{e\}}) = A(\mathcal{Y})^{-2}$$

which holds by [9] (see also [8, §2]) since \mathcal{Y} has dimension 2 and general fiber of dimension 1.

10 Harmless base changes

Lemma 10.1 *Let r be an odd prime not in S , and let N_{∞} be the cyclotomic \mathbf{Z}_r extension of \mathbf{Q} . For each prime $p \in S$, there are finitely many primes over p in N_{∞} , and each has infinite residue field. We will say that Proposition 9.1(i) is true after a harmless base change at r if it is true when \mathcal{X} is replaced by $\mathcal{X}' = \mathcal{X} \otimes_{\mathbf{Z}} O_N$ for some subfield N of N_{∞} , where O_N is the ring of integers of N . Then Proposition 9.1(i) holds if it is true after a harmless base change at two distinct primes not in S , and this implies Theorem 6.4.*

Proof: The statements about the decomposition of p in N_{∞} are clear from class field theory. To establish the rest of the Lemma, let N be a subfield of N_{∞} . Since O_N is unramified over S , one sees

that $\mathcal{X}' = \mathcal{X} \otimes_{\mathbf{Z}} O_N$ satisfies all the hypotheses of Theorem 6.4. Since \mathcal{X} is flat over \mathbf{Z} , $\mathcal{Y}' = \mathcal{X}'/G$ is isomorphic to $\mathcal{Y} \otimes_{\mathbf{Z}} O_N$. We claim that

$$\epsilon_{v,0}(\mathcal{Y}', V) = \epsilon_{v,0}(\mathcal{Y}, V)^{[N:\mathbf{Q}]} \quad (10.1)$$

if V is symplectic of dimension 0. If v is finite, this follows from the Hasse Davenport Theorem and the formulas for $\epsilon_{v,0}(\mathcal{Y}', V)$ in terms of Gauss sums which are proved by Saito in [22, Theorem 2] (see also [4, §9.4]). If v is the infinite place of \mathbf{Q} , the completions of N above v are real. From this one sees that the virtual representation ${}^1h_{v,l}(X', V)$ of the Weil-Deligne group 1W_v which is associated in [4, §2.3] to the V part of the Hodge realization of the motive of $X' = \mathbf{Q} \otimes_{\mathbf{Z}} \mathcal{X}'$ is isomorphic to the sum of $[N : \mathbf{Q}]$ copies of the corresponding motive for $X = \mathbf{Q} \otimes_{\mathbf{Z}} \mathcal{X}$. Therefore (10.1) when v is the infinite place follows from the multiplicativity of ϵ -constants.

By flat base change, $R, (\mathcal{X}', O_N \otimes_{\mathbf{Z}} \omega_{\mathcal{X},S}^{\bullet}(0)[1]) = O_N \otimes_{\mathbf{Z}} R, (\omega_{\mathcal{X},S}^{\bullet}(0)[1])$. On \mathcal{X}' we have an exact sequence of complexes

$$0 \rightarrow O_N \otimes_{\mathbf{Z}} \omega_{\mathcal{X},S}^{\bullet}(0)[1] \rightarrow \omega_{\mathcal{X}',0}^{\bullet}(0)[1] \rightarrow T \rightarrow 0$$

in which T is a complex of coherent $\mathcal{O}_{\mathcal{X}'}$ - G -modules supported on the fiber \mathcal{X}'_r of \mathcal{X}' over r . By our choice of r , the action of G on the reduction of \mathcal{X}'_r is free. Hence Propositions 7.1 and 7.2 show that

$$\chi_H(\mathcal{X}', G)(0) = d(O_N \otimes_{\mathbf{Z}} R, (\omega_{\mathcal{X},S}^{\bullet}(0)[1]), \langle, \rangle') \cdot t_p(F) \quad (10.2)$$

where F is a free $\mathbf{F}_p G$ -module and \langle, \rangle' is the perfect pairing on

$$R, (\mathcal{X}', \omega_{\mathcal{X}',0}^{\bullet}(0)[1])_{\mathbf{Q}} = R, (\mathcal{X}', O_N \otimes_{\mathbf{Z}} \omega_{\mathcal{X},S}^{\bullet}(0)[1])_{\mathbf{Q}} = N \otimes_{\mathbf{Q}} R, (\omega_{\mathcal{X},S}^{\bullet}(0)[1])_{\mathbf{Q}} \quad (10.3)$$

which is used to compute $\chi_H(\mathcal{X}', G)(0)$. Passing to characters of degree 0, we see

$$\tilde{t}_p(F) = 0. \quad (10.4)$$

The pairing \langle, \rangle' appearing in (10.2) is by duality and the isomorphisms in (10.3) the composition of $1 \otimes (\langle, \rangle)$ on $N \otimes_{\mathbf{Q}} R, (\omega_{\mathcal{X},S}^{\bullet}(0)[1])_{\mathbf{Q}}$ with the trace homomorphism $tr_{N/\mathbf{Q}} : N \rightarrow \mathbf{Q}$. Therefore, Corollary A.6 show that on characters of degree 0 we have

$$\tilde{d}(O_N \otimes_{\mathbf{Z}} R, (\omega_{\mathcal{X},S}^{\bullet}(0)[1]), \langle, \rangle') = \tilde{d}(R, (\omega_{\mathcal{X},S}^{\bullet}(0)[1]), \langle, \rangle)^{[N:\mathbf{Q}]} \quad (10.5)$$

Now combining (10.2), (10.4) and (10.5) with the definition of $\chi_H(\mathcal{X}, G)(0)$ gives

$$\tilde{\chi}_H(\mathcal{X}', G)(0) = (\tilde{\chi}_H(\mathcal{X}, G)(0))^{[N:\mathbf{Q}]} \quad (10.6)$$

Putting together (10.1) and (10.6) shows that if Theorem 6.4 is true after a harmless base change at r , then

$$\tilde{\chi}_H(\mathcal{X}, G)(0)^{[N:\mathbf{Q}]} = \tilde{\epsilon}_0^s(\mathcal{Y})^{[N:\mathbf{Q}]}$$

in $Ad HCl(\mathbf{Z}G)$ where $[N : \mathbf{Q}]$ is power of r . This shows Proposition 9.1(i) is true if is is true after harmless base changes by two different primes r . It was shown in Proposition 9.1 that this will then imply Theorem 6.4

Remark 10.2 A harmless base change at r will in general add r to the set S , since the base change $O_N \otimes \mathcal{Y}$ will not in general have smooth fiber over r . However, the fiber of $O_N \otimes \mathcal{Y}$ over the unique prime of O_N above r will be smooth over the residue field of this prime.

11 Some constructions on surfaces.

In this section we will suppose Hypothesis 6.2. We first need to generalize the discriminant $\chi_H(\mathcal{X}, G)(0)$ defined in (6.6).

Suppose \mathcal{J} is a Weil divisor on \mathcal{Y} , and let $\mathcal{J}' = \pi^{-1}(\mathcal{J})$. Define $\omega_{\mathcal{X}, S}^\bullet(\mathcal{J}')[1]$ to be the complex $[\mathcal{O}_{\mathcal{X}}(-\mathcal{J}') \rightarrow \omega_{\mathcal{X}}(\mathcal{X}_S^{\text{red}} + \mathcal{J}')]]$ having terms in degree -1 and 0 and trivial boundary homomorphism. One has

$$H^j(\omega_{\mathcal{X}, S}^\bullet(\mathcal{J}')[1]) = \begin{cases} H^0(\mathcal{O}_{\mathcal{X}}(-\mathcal{J}')) & \text{if } j = -1 \\ H^1(\mathcal{O}_{\mathcal{X}}(-\mathcal{J}')) \oplus H^0(\omega_{\mathcal{X}}(\mathcal{X}_S^{\text{red}} + \mathcal{J}')) & \text{if } j = 0 \\ H^1(\omega_{\mathcal{X}}(\mathcal{X}_S^{\text{red}} + \mathcal{J}')) & \text{if } j = 1 \\ 0 & \text{otherwise} \end{cases} \quad (11.1)$$

Let J (resp. J') be the intersection of \mathcal{J} (resp. \mathcal{J}') with the general fiber Y (resp. X) of \mathcal{Y} (resp. \mathcal{X}). The restriction $\omega_{\mathcal{X}, S}^\bullet(\mathcal{J}')[1]_{\mathbf{Q}}$ of $\omega_{\mathcal{X}, S}^\bullet(\mathcal{J}')[1]$ to X is the complex $[\mathcal{O}_X(-J') \rightarrow \Omega_{X/\mathbf{Q}}^1(J')]]$ with terms in degree -1 and 0 and trivial boundary homomorphism.

Define G -equivariant non-degenerate pairings

$$\langle \cdot, \cdot \rangle_t : H^t(R, (\omega_{\mathcal{X}, S}^\bullet(\mathcal{J}')[1])_{\mathbf{Q}}) \times H^{-t}(R, (\omega_{\mathcal{X}, S}^\bullet(\mathcal{J}')[1])_{\mathbf{Q}}) \rightarrow \mathbf{Q} \quad (11.2)$$

in the following way. If $t = 0$, then $H^0(R, (\omega_{\mathcal{X}, S}^\bullet(\mathcal{J}')[1])_{\mathbf{Q}}) = H^1(X, \mathcal{O}_X(-J')) \oplus H^0(X, \Omega_{X/\mathbf{Q}}^1(J'))$. Let

$$\langle \cdot, \cdot \rangle_0 = \langle \cdot, \cdot \rangle_{1,0} + \langle \cdot, \cdot \rangle_{0,1} \quad (11.3)$$

where $\langle \cdot, \cdot \rangle_{i,1-i}$ is the duality pairing

$$\langle \cdot, \cdot \rangle_{i,j} : H^i(X, \Omega_X^j((2j-1)J')) \times H^{1-i}(X, \Omega_X^{1-j}((1-2j)J')) \rightarrow \mathbf{Q}.$$

If $t = -1$, then

$$H^{-1}(R, (\omega_{\mathcal{X}, S}^\bullet(\mathcal{J}')[1])_{\mathbf{Q}}) = H^0(X, \mathcal{O}_X(-J'))$$

and

$$H^1(R, (\omega_{\mathcal{X}, S}^\bullet(\mathcal{J}')[1])_{\mathbf{Q}}) = H^1(X, \Omega_{X/\mathbf{Q}}^1(J')).$$

Define the pairing $\langle \cdot, \cdot \rangle_{-1}$ to be the duality pairing $\langle \cdot, \cdot \rangle_{0,0}$, and let $\langle \cdot, \cdot \rangle_1 = \langle \cdot, \cdot \rangle_{1,1}$. If $t < -1$ or $t > 1$, let $\langle \cdot, \cdot \rangle_t$ be the trivial pairing. The resulting system of pairings $\langle \cdot, \cdot \rangle = \{\langle \cdot, \cdot \rangle_t\}_t$ is perfect in the sense of Definition 2.6 (see also Definition 3.3). Let

$$\chi_H(\mathcal{X}, G)(\mathcal{J}') = d(R, (\omega_{\mathcal{X}, S}^\bullet(\mathcal{J}')[1]), \langle \cdot, \cdot \rangle) \quad (11.4)$$

be the associated discriminant. When $\mathcal{J}' = 0$, this is the discriminant defined in (6.6).

The following result is proved in much the same way as [4, Proposition 9.1.3], so we will omit the proof.

Lemma 11.1 *Let \mathcal{J} be a very ample divisor on \mathcal{Y} which is a positive integral combination of horizontal divisors; such \mathcal{J} exist by [17]. After making a harmless base change and replacing \mathcal{J} by a divisor linear equivalent to a positive integral multiple of \mathcal{J} , we can assume that there are divisors \mathcal{D} and \mathcal{F} on \mathcal{Y} with the following properties.*

- a. The divisor \mathcal{F} is an integral combination of irreducible components of fibers of \mathcal{Y} over $p \notin S$.
- b. There is a canonical divisor $K_{\mathcal{Y}}$ on \mathcal{Y} such that

$$K_{\mathcal{Y}} + \mathcal{Y}_S^{\text{red}} + 2\mathcal{J} + \mathcal{F} = \mathcal{D}$$

where $\mathcal{Y}_S^{\text{red}}$ is the sum of the reductions $\mathcal{Y}_p^{\text{red}}$ of the fibers of \mathcal{Y} over the primes p in S .

- c. The divisors \mathcal{J} and \mathcal{D} are effective, very ample and positive integral combinations of horizontal divisors.
- d. For $p \in S$, let $\mathcal{E} = \mathcal{D} + \mathcal{J}$. Then \mathcal{E} intersects $\mathcal{Y}_p^{\text{red}}$ transversely at distinct smooth points of $\mathcal{Y}_p^{\text{red}}$. Suppose y is such a point. Then y is a regular point of \mathcal{E} . The scheme-theoretic inverse image $\pi^{-1}(\mathcal{E})$ of \mathcal{E} with respect to the quotient morphism $\pi : \mathcal{X} \rightarrow \mathcal{Y}$ is a sum of distinct irreducible horizontal divisors on \mathcal{X} , and $\pi^{-1}(\mathcal{E})$ intersects the reduced fiber $\mathcal{X}_p^{\text{red}}$ at smooth points and this intersection is transversal. Each point of x of $\pi^{-1}(y)$ is regular on $\pi^{-1}(\mathcal{E})$, and $\mathcal{O}_{\pi^{-1}(\mathcal{E}),x}/\mathcal{O}_{\mathcal{E},y}$ is a tame extension of discrete valuation rings.

Definition 11.2 *With the notations of Lemma 11.1, let s be a global section of the line bundle $\omega_{\mathcal{Y}}(\mathcal{Y}_S^{\text{red}} + 2\mathcal{J} + \mathcal{F})$ having divisor \mathcal{D} .*

On \mathcal{Y} we have an exact sequence

$$0 \rightarrow \mathcal{O}_{\mathcal{Y}} \xrightarrow{-s} \omega_{\mathcal{Y}}(\mathcal{Y}_S^{\text{red}} + 2\mathcal{J} + \mathcal{F}) \rightarrow \omega_{\mathcal{Y}}(\mathcal{Y}_S^{\text{red}} + 2\mathcal{J} + \mathcal{F})|_{\mathcal{D}} \rightarrow 0. \quad (11.5)$$

By Lemma 9.2 we have

$$\mathcal{O}_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{Y}}} \omega_{\mathcal{Y}}(\mathcal{Y}_S^{\text{red}}) = \pi^*(\omega_{\mathcal{Y}}(\mathcal{Y}_S^{\text{red}})) = \omega_{\mathcal{X}}(\mathcal{X}_S^{\text{red}}).$$

Define $\mathcal{D}' = \pi^{-1}(\mathcal{D})$, $\mathcal{J}' = \pi^{-1}(\mathcal{J})$ and $\mathcal{F}' = \pi^{-1}(\mathcal{F})$. On tensoring (11.5) with $\mathcal{O}_{\mathcal{X}}$ we arrive at an exact sequence

$$0 \rightarrow \mathcal{O}_{\mathcal{X}} \xrightarrow{-s} \omega_{\mathcal{X}}(\mathcal{X}_S^{\text{red}} + 2\mathcal{J}' + \mathcal{F}') \rightarrow \omega_{\mathcal{X}}(\mathcal{X}_S^{\text{red}} + 2\mathcal{J}' + \mathcal{F}')|_{\mathcal{D}'} \rightarrow 0.$$

Twisting by $\pi^{-1}\mathcal{O}_{\mathcal{Y}}(-\mathcal{J}) = \mathcal{O}_{\mathcal{X}}(-\mathcal{J}')$ now gives an exact sequence

$$0 \rightarrow \mathcal{O}_{\mathcal{X}}(-\mathcal{J}') \xrightarrow{-s} \omega_{\mathcal{X}}(\mathcal{X}_S^{\text{red}} + \mathcal{J}' + \mathcal{F}') \rightarrow \omega_{\mathcal{X}}(\mathcal{X}_S^{\text{red}} + \mathcal{J}' + \mathcal{F}')|_{\mathcal{D}'} \rightarrow 0. \quad (11.6)$$

Lemma 11.3 *After a harmless base change, we can assume without loss of generality that*

$$H^0(\mathcal{O}_{\mathcal{X}}(-\mathcal{J}')) = 0 = H^1(\omega_{\mathcal{X}}(\mathcal{J}')) = H^1(\omega_{\mathcal{X}}(\mathcal{X}_S^{\text{red}} + \mathcal{J}' + \mathcal{F}')) \quad (11.7)$$

and that the long exact cohomology sequence of (11.6) gives a short exact sequence

$$0 \rightarrow H^0(\omega_{\mathcal{X}}(\mathcal{X}_S^{\text{red}} + \mathcal{J}' + \mathcal{F}')) \rightarrow H^0(\omega_{\mathcal{X}}(\mathcal{X}_S^{\text{red}} + \mathcal{J}' + \mathcal{F}')|_{\mathcal{D}'}) \rightarrow H^1(\mathcal{O}_{\mathcal{X}}(-\mathcal{J}')) \rightarrow 0 \quad (11.8)$$

of finitely generated projective $\mathbf{Z}G$ -modules. The modules

$$H^0(\omega_{\mathcal{X}}(\mathcal{X}_S^{\text{red}} + \mathcal{J}')) \quad \text{and} \quad H^0(\omega_{\mathcal{X}}(\mathcal{X}_S^{\text{red}} + \mathcal{J}' + \mathcal{F}'))$$

are projective $\mathbf{Z}G$ -modules of maximal rank inside $H^0(\omega_{\mathcal{X}}(\mathcal{J}'))$, and their generalized index in $K_0T(\mathbf{Z}G)$ is in the image of the induction homomorphism $K_0T(\mathbf{Z}) \rightarrow K_0T(\mathbf{Z}G)$.

Proof: Since \mathcal{J} is a non-zero horizontal effective divisor on \mathcal{Y} , one has $H^0(\mathcal{O}_{\mathcal{X}}(-\mathcal{J}')) = 0$. Since $\pi : \mathcal{X} \rightarrow \mathcal{Y}$ is finite, we have for $\mathcal{L} = \omega_{\mathcal{X}}$ and $\mathcal{L} = \omega_{\mathcal{X}}(\mathcal{X}_S^{red})$ that

$$H^1(\mathcal{L}(\mathcal{J}')) = H^1(\pi_*(\mathcal{L}(\mathcal{J}'))) = H^1(\pi_*(\mathcal{L}) \otimes \mathcal{O}_{\mathcal{Y}}(\mathcal{J})).$$

We were free in Lemma 11.1 to initially pick \mathcal{J} initially to be a very ample effective horizontal divisor on \mathcal{Y} . By theorem of Serre (c.f. [18, Thm. III.5.2]) for each such \mathcal{J} there is an integer n_0 so $H^1(\pi_*(\mathcal{L}) \otimes \mathcal{O}_{\mathcal{Y}}(n\mathcal{J})) = 0$ for $\mathcal{L} = \omega_{\mathcal{X}}$ and $\mathcal{L} = \omega_{\mathcal{X}}(\mathcal{X}_S^{red})$ and all $n \geq n_0$. Replacing \mathcal{J} by $n_0\mathcal{J}$, we can assume $n_0 = 1$. Lemma 11.1 now involves a harmless base change and a replacement of \mathcal{J} by a divisor linearly equivalent to a positive integral multiple of \mathcal{J} . This base change may change S as in Remark 10.2, and the pullback of $\omega_{\mathcal{X}}$ via the base change may need to be adjusted by a divisor over the prime r involved in the base change. However, after the base change, we will have

$$H^0(\mathcal{O}_{\mathcal{X}}(-\mathcal{J}')) = 0 = H^1(\omega_{\mathcal{X}}(\mathcal{J}' + \mathcal{T}'_1)) = H^1(\omega_{\mathcal{X}}(\mathcal{J}' + \mathcal{X}_S^{red} + \mathcal{T}'_2)) \quad (11.9)$$

for some fibral divisors \mathcal{T}'_1 and \mathcal{T}'_2 supported over the prime r . We wish to show that this implies

$$H^1(\omega_{\mathcal{X}}(\mathcal{J}')) = H^1(\omega_{\mathcal{X}}(\mathcal{J}' + \mathcal{X}_S^{red} + \mathcal{F}')) = 0. \quad (11.10)$$

Since \mathcal{Y} is regular and equidimensional, we can reduce to the case in which \mathcal{Y} is irreducible. Let O_K be the integral closure of \mathbf{Z} in the function field of \mathcal{Y} , so that \mathcal{Y} is a flat scheme over O_K . Suppose \mathcal{T} is any integral combination of irreducible components of smooth fibers of \mathcal{Y} over O_K , and that \mathcal{E} is a line bundle on \mathcal{X} which is an $O_{\mathcal{X}}\text{-}G$ -module. Then $\mathcal{O}_{\mathcal{X}}(\pi^{-1}(\mathcal{T})) = (\pi \circ \zeta)^{-1}(L)$ for some line bundle L on $\text{Spec}(O_K)$, where $\zeta : \mathcal{Y} \rightarrow \text{Spec}(O_K)$ is the natural morphism. Hence

$$H^j(\mathcal{E} \otimes O_{\mathcal{X}}(\pi^{-1}(\mathcal{T}))) = H^j(\mathcal{E}) \otimes_{O_K} H^0(L) \quad (11.11)$$

for all j by flat base change. Letting $\mathcal{E} = \omega_{\mathcal{X}}(\mathcal{J}')$ and $\mathcal{E} = \omega_{\mathcal{X}}(\mathcal{J}' + \mathcal{X}_S^{red})$ we now deduce (11.10) from (11.9) and (11.11).

The exact sequence (11.8) is now clear from (11.6) and (11.7). Since the action of G on \mathcal{X} is tame, we know that $R, (\omega_{\mathcal{X}}(\mathcal{X}_S^{red} + \mathcal{J}' + \mathcal{F}')), R, (\omega_{\mathcal{X}}(\mathcal{X}_S^{red} + \mathcal{J}' + \mathcal{F}')|_{\mathcal{D}'})$ and $R, (\mathcal{O}_{\mathcal{X}}(-\mathcal{J}'))$ are perfect complexes of $\mathbf{Z}G$ -modules. Since we have shown these sheaves to have at most one non-zero cohomology group, it follows that this cohomology group must be of finite projective dimension for $\mathbf{Z}G$. Since $H^0(\omega_{\mathcal{X}}(\mathcal{X}_S^{red} + \mathcal{J}' + \mathcal{F}'))$ has no \mathbf{Z} -torsion because \mathcal{X} is flat over \mathbf{Z} , we see that $H^0(\omega_{\mathcal{X}}(\mathcal{X}_S^{red} + \mathcal{J}' + \mathcal{F}'))$ must be a projective $\mathbf{Z}G$ -module. We now show that $H^1(\mathcal{O}_{\mathcal{X}}(-\mathcal{J}'))$ has no non-trivial \mathbf{Z} -torsion, since then $H^1(\mathcal{O}_{\mathcal{X}}(-\mathcal{J}'))$ will be forced to be projective, and $H^0(\omega_{\mathcal{X}}(\mathcal{X}_S^{red} + \mathcal{J}' + \mathcal{F}')|_{\mathcal{D}'})$ will be as well by the exactness of (11.8).

The duality theorem for \mathcal{X} (c.f. [19, Theorem 11.1]) implies that the \mathbf{Z} -torsion in the finitely generated abelian group $H^1(\mathcal{O}_{\mathcal{X}}(-\mathcal{J}'))$ is Pontryagin dual to the \mathbf{Z} -torsion in $H^1(\omega_{\mathcal{X}}(\mathcal{J}'))$. This is not a misprint; the co-torsion of $H^1(\mathcal{O}_{\mathcal{X}}(-\mathcal{J}'))$ is \mathbf{Z} -dual to the torsion-free group $H^0(\omega_{\mathcal{X}}(\mathcal{J}'))$. Since we have arranged in (11.7) that $H^1(\omega_{\mathcal{X}}(\mathcal{J}')) = 0$, we conclude that $H^1(\mathcal{O}_{\mathcal{X}}(-\mathcal{J}'))$ has no non-trivial \mathbf{Z} -torsion.

All that remains to be shown is the last assertion of the Lemma concerning $H^0(\omega_{\mathcal{X}}(\mathcal{X}_S^{red} + \mathcal{J}'))$ and $H^0(\omega_{\mathcal{X}}(\mathcal{X}_S^{red} + \mathcal{J}' + \mathcal{F}'))$. We have shown $H^0(\omega_{\mathcal{X}}(\mathcal{X}_S^{red} + \mathcal{J}' + \mathcal{F}'))$ is a projective $\mathbf{Z}G$ -module, and this module is a \mathbf{Z} -lattice of maximal rank in the \mathbf{Q} vector space $H^0(\omega_{\mathcal{X}}(\mathcal{J}'))$. Hence the last assertion of Lemma 11.3 follows from (11.11) after reducing to the case in which \mathcal{Y} is irreducible.

Corollary 11.4 *If \mathcal{T} is a scheme over \mathbf{Z} , let T in Roman type denote $\mathbf{Q} \otimes_{\mathbf{Z}} \mathcal{T}$. The sequence (11.6) determines on X the exact sequence*

$$0 \longrightarrow \mathcal{O}_X(-J') \xrightarrow{-s} \omega_X(J') \longrightarrow \omega_X(J')|_{D'} \longrightarrow 0 \quad (11.12)$$

where $\omega_X = \Omega_X^1/\mathbf{Q}$. The long exact cohomology sequence associated to (11.12) is the following exact sequence of free $\mathbf{Q}G$ -modules:

$$0 \longrightarrow H^0(\omega_X(J')) \xrightarrow{-\alpha} H^0(\omega_X(J')|_{D'}) \xrightarrow{\beta} H^1(\mathcal{O}_X(-J')) \longrightarrow 0. \quad (11.13)$$

This sequence results from tensoring (11.8) with \mathbf{Q} .

Our goal now is to define a symmetric bilinear G -equivariant non-degenerate bilinear form on the middle term $H^0(\omega_X(J')|_{D'})$ in (11.13). The scheme D' is a sum of closed points on X ; let $\omega_{D'}$ be the relative dualizing sheaf of D' over \mathbf{Q} . We will need to make explicit the following sequence of isomorphisms:

$$(\omega_X(J')|_{D'})^{\otimes 2} = (\omega_X(J')|_{D'}) \otimes \mathcal{O}_X(K_X + J')|_{D'} \quad (11.14)$$

$$= \omega_X(K_X + 2 \cdot J')|_{D'} \quad (11.15)$$

$$= \omega_X(D')|_{D'} \quad (11.16)$$

$$= \omega_{D'} \quad (11.17)$$

The canonical divisor K_X is

$$K_X = \mathbf{Q} \otimes_{\mathbf{Z}} (K_X + \mathcal{X}_S^{red} + \mathcal{F}') = \mathbf{Q} \otimes_{\mathbf{Z}} \pi^{-1}(K_Y + \mathcal{Y}_S^{red} + \mathcal{F}) = \mathbf{Q} \otimes_{\mathbf{Z}} \pi^{-1}(\mathcal{D} - 2\mathcal{J}) = D' - 2J'. \quad (11.18)$$

Thus $K_X + J' = D' - J'$. From [14, p. 136] we have an isomorphism

$$\mathcal{O}_X(K_X + J') \rightarrow \omega_X(J') \quad (11.19)$$

sending a meromorphic section β of $\mathcal{O}_X(K_X + J')$ to $\beta \cdot s$, where

$$s \in (\mathcal{Y}, \omega_{\mathcal{Y}}(\mathcal{Y}_S^{red} + 2\mathcal{J} + \mathcal{F})) \subset (\mathcal{Y}, \omega_{\mathcal{Y}}(2\mathcal{J})) \subset (X, \omega_X(2J'))$$

is the global section specified in Definition 11.2. We thus have an isomorphism

$$(\omega_X(J'))^{\otimes 2} \rightarrow \omega_X(J') \otimes \mathcal{O}_X(K_X + J') = \omega_X(K_X + 2 \cdot J') \quad (11.20)$$

in which $(\beta_1 s) \otimes (\beta_2 s)$ goes to $\beta_1 \cdot \beta_2 \cdot s$. Note that this isomorphism is symmetric, in the sense that $(\beta_2 s) \otimes (\beta_1 s)$ also has image $\beta_1 \cdot \beta_2 \cdot s$. It is also G -equivariant, in the sense that $g(\beta_1 s) \otimes g(\beta_2 s) = (g(\beta_1) s) \otimes (g(\beta_2) s)$ goes to $g(\beta_1 \cdot \beta_2) \cdot s$ for $g \in G$, since s is fixed by G . The isomorphism in (11.20) gives (11.14) and (11.15), while (11.16) comes from (11.18). Finally, (11.17) is the adjunction formula.

Proposition 11.5 *The isomorphisms in (11.14) - (11.17) and the canonical trace homomorphism $Tr : H^0(D', \omega_{D'}) \rightarrow \mathbf{Q}$ give a non-degenerate symmetric G -equivariant bilinear form*

$$\langle, \rangle : H^0(\omega_X(J')|_{D'}) \times H^0(\omega_X(J')|_{D'}) \rightarrow \mathbf{Q}. \quad (11.21)$$

Suppose $a \in H^0(\omega_X(J'))$, $b \in H^0(\omega_X(J')|_{D'})$ and that α and β are the homomorphisms in (11.13). Let $\cup : H^0(\omega_X(J')) \times H^1(\mathcal{O}_X(-J')) \rightarrow H^1(\omega_X)$ be the tensor product pairing, and let $tr : H^1(\omega_X) \rightarrow \mathbf{Q}$ be the trace isomorphism. Then

$$\langle \alpha(a), b \rangle = \langle a, \beta(b) \rangle' \quad (11.22)$$

where

$$\langle, \rangle' : H^0(\omega_X(J')) \times H^1(\mathcal{O}_X(-J')) \rightarrow \mathbf{Q} \quad (11.23)$$

is the perfect G -equivariant duality pairing defined by $\langle a, m \rangle' = tr(a \cup m)$ for $a \in H^0(\omega_X(J'))$ and $m \in H^1(\mathcal{O}_X(-J'))$.

Proof: The pairing (11.21) is symmetric and G -equivariant because the pairing in (11.20) has these properties. Suppose we show that (11.22) holds. Then since \langle, \rangle' in (11.23) is perfect and G -equivariant by Serre duality, it will follow that \langle, \rangle is non-degenerate.

Observe first that we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_X(-J') & \xrightarrow{i} & \mathcal{O}_X(K_X + J') & \longrightarrow & \mathcal{O}_X(K_X + J')|_{D'} \longrightarrow 0 \\ & & \downarrow & & \cdot s \downarrow & & \psi \downarrow \\ 0 & \longrightarrow & \mathcal{O}_X(-J') & \xrightarrow{\cdot s} & \omega_X(J') & \longrightarrow & \omega_X(J')|_{D'} \longrightarrow 0 \end{array} \quad (11.24)$$

in which the left downward arrow is the identity map, the middle down arrow is the isomorphism given by multiplication by s , and the right down arrow is the induced isomorphism. The map i in the top row is the natural injection coming from the fact that $D' = K_X + 2J'$ is effective.

To compute $\langle \alpha(a), b \rangle$ we now unravel (11.14) - (11.17). From (11.19), $a \in H^0(\omega_X(J'))$ has the form $a = a' \cdot s$ for some $a' \in H^0(\mathcal{O}_X(K_X + J'))$. Let \mathcal{F}_x be the stalk at $x \in X$ of a coherent sheaf \mathcal{F} on X . Choose $u_x \in \mathcal{O}_X(K_X + J')_x$ for $x \in D'$ so $\psi(\sum_{x \in D'} u_x) = b$ in the right column of (11.24). Fix $x \in D'$. In (11.20) we need to find

$$a' \cdot u_x \cdot s \in \omega_X(K_X + 2 \cdot J')_x = \omega_X(D')_x.$$

The adjunction formula comes about from the residue homomorphism

$$res_x : \omega_X(D')_x \rightarrow \mathbf{Q}(x)$$

where $\mathbf{Q}(x)$ is the residue field of the point $x \in D'$. We have a trace homomorphism $tr_x : \mathbf{Q}(x) \rightarrow \mathbf{Q}$. Putting together all these ingredients and the definition of \langle, \rangle in (11.21) gives the formula

$$\langle \alpha(a), b \rangle = \sum_{x \in D'} tr_x(res_x(a' \cdot u_x \cdot s)). \quad (11.25)$$

We now have to compute the cup product pairing $tr(a \cup \beta(b))$. Let $j : \eta \rightarrow X$ be the inclusion of the generic point η of X into X . The top row of (11.24) fits into a diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{O}_X(-J') & \xrightarrow{i} & \mathcal{O}_X(K_X + J') & \longrightarrow & \mathcal{O}_X(K_X + J')|_{D'} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{O}_X(-J') & \longrightarrow & j_*\mathcal{O}_{X,\eta} & \longrightarrow & \bigoplus_{x \in X^0} \frac{j_*\mathcal{O}_{X,\eta}}{\mathcal{O}_X(-J')_x} \longrightarrow 0
\end{array} \tag{11.26}$$

in which X^0 is the set of closed points of X , and all the homomorphisms are the obvious inclusions. The tensor product of this diagram with $\omega_X(J')$ is a diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \omega_X & \longrightarrow & \omega_X(D') & \longrightarrow & \bigoplus_{x \in D'} \frac{\omega_X(D')_x}{\omega_{X,x}} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \omega_X & \longrightarrow & j_*\omega_{X,\eta} & \longrightarrow & \bigoplus_{x \in X^0} \frac{j_*\omega_{X,\eta}}{\omega_{X,x}} \longrightarrow 0
\end{array} \tag{11.27}$$

The bottom row of (11.27) is used to compute the trace isomorphism $tr : H^1(\omega_X) \rightarrow \mathbf{Q}$ in the following way (c.f. [18, p. 248]). The cohomology of the bottom row of (11.27) gives an exact sequence

$$\omega_{X,\eta} \longrightarrow \bigoplus_{x \in X^0} \frac{\omega_{X,\eta}}{\omega_{X,x}} \xrightarrow{\delta} H^1(\omega_X) \longrightarrow 0. \tag{11.28}$$

Given a set of elements $z_x \in \omega_{X,\eta}$ such that $z_x \in \omega_{X,x}$ for all but finitely many x , one has

$$tr(\delta(\sum_{x \in X^0} [z_x])) = \sum_{x \in X^0} tr_x(res_x(z_x)) \tag{11.29}$$

where $[z_x]$ is the image of z_x in $\omega_{X,\eta}/\omega_{X,x}$.

We now put this together to compute $tr(a \cup \beta(b))$. Recall that $u_x \in \mathcal{O}_X(K_X + J')_x$ was chosen so that $\bigoplus_{x \in D'} u_x$ has image $b \in \omega_X(J')|_{D'}$ under the right downwards homomorphism ψ in (11.24). Since $a \in H^0(\omega_X(J'))$, and tensoring with $\omega_X(J')$ takes (11.26) to (11.27), we see that

$$\delta(\bigoplus_{x \in D'} [u_x \cdot a]) = a \cup \beta(b) \quad \text{in } H^1(\omega_X) \tag{11.30}$$

where δ is the boundary homomorphism in (11.28). Accordingly, (11.29) gives

$$tr(a \cup \beta(b)) = \sum_{x \in X^0} tr_x(res_x(u_x \cdot a)). \tag{11.31}$$

The fact that (11.31) has the same value as (11.25) now results from $a = a' \cdot s$.

Corollary 11.6 *The projective $\mathbf{Z}G$ -module $H^0(\omega_{\mathcal{X}}(\mathcal{X}_S^{red} + \mathcal{J}' + \mathcal{F}')|_{\mathcal{D}'})$ together with the pairing \langle, \rangle defined in Proposition 11.5 defines Hermitian $\mathbf{Z}G$ -module. For $n \in \mathbf{Z}$, let $(M_n(\mathcal{J}'), \langle, \rangle_0)$ be the Hermitian module*

$$M_n(\mathcal{J}') = H^1(\mathcal{O}_{\mathcal{X}}(-\mathcal{J}')) \bigoplus H^0(\omega_{\mathcal{X}}(\mathcal{X}_S^{red} + \mathcal{J}' + n\mathcal{F}')) \quad (11.32)$$

with \langle, \rangle_0 the pairing on $M_n(\mathcal{J}')_{\mathbf{Q}}$ described in (11.2) and (11.3) of §6. We then have an equality of discriminants

$$d(H^0(\omega_{\mathcal{X}}(\mathcal{X}_S^{red} + \mathcal{J}' + \mathcal{F}')|_{\mathcal{D}'}, \langle, \rangle)) = d(M_1(\mathcal{J}'), \langle, \rangle_0) = d(R, (\omega_{\mathcal{X}, S}^{\bullet}(\mathcal{J}')[1]), \langle, \rangle) \quad (11.33)$$

where the last term is defined in (11.4). We have an equality of degree 0 discriminants

$$\tilde{d}(M_1(\mathcal{J}'), \langle, \rangle_0) = \tilde{d}(M_0(\mathcal{J}'), \langle, \rangle_0). \quad (11.34)$$

Proof: To show the first equality in (11.33) we apply Lemma B.1 to the sequence (11.13) and we use the relation 11.22. The second equality in (11.33) is clear from the calculation of the pairing \langle, \rangle just before (11.4) and from the fact that $H^j(\omega_{\mathcal{X}, S}^{\bullet}(\mathcal{J}')[1]) = 0$ if $j \neq 0$. Finally, (11.34) results from the comparison of $H^0(\omega_{\mathcal{X}}(\mathcal{X}_S^{red} + \mathcal{J}' + \mathcal{F}'))$ and $H^0(\omega_{\mathcal{X}}(\mathcal{X}_S^{red} + \mathcal{J}'))$ at the end of Lemma 11.3. This is because the restriction to characters of degree 0 of a class in $HCl(\mathbf{Z}G)$ which is the image of class in the image of the induction homomorphism $K_0T(\mathbf{Z}) \rightarrow K_0T(\mathbf{Z}G)$ is trivial.

Our goal now is to compute $d(H^0(\omega_{\mathcal{X}}(\mathcal{X}_S^{red} + \mathcal{J}' + \mathcal{F}')|_{\mathcal{D}'}, \langle, \rangle))$ in $HCl(\mathbf{Z}G)$ in a different way, via symplectic root numbers.

Fix $y \in D$. By Lemma 11.1(d), D and J are disjoint and reduced, since they have disjoint reduced specializations to a finite fiber of \mathcal{Y} . Hence a uniformizing parameter z_y at y on Y is a local equation for D , and 1 is a local equation for J . We fix an isomorphism of stalks $\iota_y : \omega_Y(J)_y = \omega_{Y, y} \rightarrow \mathcal{O}_{Y, y}$ by requiring $\iota_y(d(z_y)) = 1$, where $d(z_y)$ is the differential of z_y . Tensoring with the residue field $k(y)$ of y over $\mathcal{O}_{Y, y}$ gives an induced isomorphism $\bar{\iota}_y : \omega_Y(J)|_y \rightarrow k(y)$. Then tensoring with \mathcal{O}_X over \mathcal{O}_Y then gives an isomorphism $\kappa_y : \omega_X(J')|_{\pi^{-1}(y)} \rightarrow \mathcal{O}_{\pi^{-1}(y)}$. Here $\pi^{-1}(y)$ is a finite union of closed points; let x be a point of X over y . Because $\pi : X \rightarrow Y$ is étale, z_y is a uniformizer in $\mathcal{O}_{X, x}$, and 1 and z_y are local equations in $\mathcal{O}_{X, x}$ for J' and \mathcal{D}' , respectively. Summing over $y \in D$ gives an isomorphism

$$\kappa = \sum_{y \in D} \kappa_y : H^0(\omega_X(J')|_{\mathcal{D}'}) = \bigoplus_{y \in D} \omega_X(J')|_{\pi^{-1}(y)} \rightarrow H^0(\mathcal{O}_{\mathcal{D}'}) = \bigoplus_{y \in D} \mathcal{O}_{\pi^{-1}(y)}. \quad (11.35)$$

Recall that $s \in H^0(Y, \omega_Y(2J))$ has divisor $D = K_Y + 2J$. Therefore there is a unit $v_y \in \mathcal{O}_{Y, y}^*$ such that

$$z_y^{-1}s = v_y \cdot d(z_y) \quad \text{in} \quad \omega_{Y, y} = \omega_Y(2J)_y. \quad (11.36)$$

Lemma 11.7 *There is a direct sum decomposition*

$$H^0(\omega_X(J')|_{\mathcal{D}'}) = \bigoplus_{y \in D} H^0(\omega_X(J')|_{\pi^{-1}(y)}). \quad (11.37)$$

Let \langle, \rangle_y be the restriction to the summand indexed by y on the right of the pairing \langle, \rangle defined in Proposition 11.5. Then (11.37) gives an orthogonal direct sum decomposition of $H^0(\omega_X(J')|_{D'})$ as an Hermitian $\mathbf{Q}G$ -module. Let $\langle, \rangle_y^\dagger : H^0(\mathcal{O}_{\pi^{-1}(y)}) \times H^0(\mathcal{O}_{\pi^{-1}(y)}) \rightarrow \mathbf{Q}$ be the natural trace pairing. Then for $a, b \in H^0(\omega_X(J')|_{\pi^{-1}(y)})$ one has

$$\langle \bar{v}_y a, b \rangle_y = \langle \kappa_y(a), \kappa_y(b) \rangle_y^\dagger \quad (11.38)$$

where \bar{v}_y is the image in $k(y) \subset \mathcal{O}_{\pi^{-1}(y)} = \bigoplus_{x \in \pi^{-1}(y)} k(x)$ of $v_y \in \mathcal{O}_{Y,y}$.

Proof: We first compute $\langle a, b \rangle$ by the method used in (11.25). Choose $t_x, u_x \in \mathcal{O}_X(K_X + J')_x$ so that $\psi(\bigoplus_{x \in D'} t_x) = a$ and $\psi(\bigoplus_{x \in D'} u_x) = b$ in the right column of (11.24). Then the same argument used to show (11.25) proves

$$\langle a, b \rangle = \sum_{x \in D'} \text{tr}_x(\text{res}_x(t_x \cdot u_x \cdot s)). \quad (11.39)$$

To analyze the right hand side of (11.39), recall that $K_X + J' = \pi^{-1}(K_Y + 2J - J) = \pi^{-1}(D - J) = D' - J'$. For $x \in D'$, the component a_x of $a \in H^0(\omega_X(J')|_{D'}) = \bigoplus_{x \in D'} \omega_X(J')|_x$ at the summand corresponding to x is the image of $t_x \cdot s \in \omega_X(J')_x$ under the natural isomorphism

$$\frac{\omega_X(J')_x}{\mathcal{M}_{X,x} \omega_X(J')_x} \rightarrow \omega_X(J')|_x$$

where $\mathcal{M}_{X,x}$ is the maximal ideal of $\mathcal{O}_{X,x}$. In view of (11.36), we have

$$t_x \cdot s = t_x \cdot z_y \cdot v_y \cdot d(z_y) \quad (11.40)$$

where $y = \pi(x)$. Similarly,

$$u_x \cdot s = u_x \cdot z_y \cdot v_y \cdot d(z_y). \quad (11.41)$$

Since $\text{res}_x(z_y^{-1} d(z_y)) = 1$, we find from (11.39) that

$$\langle a, b \rangle = \sum_{x \in D'} \text{tr}_x(\text{res}_x(t_x \cdot u_x \cdot z_y \cdot v_y \cdot d(z_y))) = \sum_{x \in D'} \text{tr}_x(r_x(t_x \cdot u_x \cdot z_y^2 \cdot v_y)) \quad (11.42)$$

where $r_x : \mathcal{O}_{X,x} \rightarrow k(x)$ is the reduction homomorphism. Observe that since $t_x, u_x \in \mathcal{O}_X(D' - J')_x$, z_y is a uniformizer in $\mathcal{O}_{X,x}$, $v_y \in \mathcal{O}_{Y,y}^*$ and 1 is a local equation for J' , the right hand side of (11.42) is well defined. Formula (11.42) shows that \langle, \rangle is the orthogonal sum over $y \in D$ of \langle, \rangle_y .

We must now compute the right hand side of (11.38). Fix $y \in D$, and suppose a and b are in the summand $H^0(\omega_X(J')|_{\pi^{-1}(y)})$ of $H^0(\omega_X(J')|_{D'})$. We fixed an isomorphism of stalks $\iota_y : \omega_Y(J)_y = \omega_{Y,y} \rightarrow \mathcal{O}_{Y,y}$ defined by $\iota_y(d(z_y)) = 1$. Since $x \in \pi^{-1}(y)$ and $X \rightarrow Y$ is étale, this gives an isomorphism of stalks $\iota_x : \omega_X(J')_x = \omega_{X,x} \rightarrow \mathcal{O}_{X,x}$ with $\iota_x(d(z_y)) = 1$. Let $\bar{\iota}_x : \omega_X(J')_x \rightarrow k(x)$ be the reduction of ι_x modulo $\mathcal{M}_{X,x}$. Under $\kappa_y : \omega_X(J')|_{\pi^{-1}(y)} \rightarrow \mathcal{O}_{\pi^{-1}(y)}$ the class $\bigoplus_{x \in \pi^{-1}(y)} a_x$ goes to $\bigoplus_{x \in \pi^{-1}(y)} \bar{\iota}_x(t_x \cdot s)$. In view of (11.40), we find that

$$\bar{\iota}_x(t_x \cdot s) = \bar{\iota}_x(t_x \cdot z_y \cdot v_y \cdot d(z_y)) = r_x(t_x \cdot z_y \cdot v_y) \quad (11.43)$$

where $r_x : \mathcal{O}_{X,x} \rightarrow k(x)$ is, as before, the reduction homomorphism. Accordingly,

$$\begin{aligned} \langle \kappa_y(a), \kappa_y(b) \rangle_y^\dagger &= \sum_{x \in \pi^{-1}(y)} \text{tr}_x(r_x(t_x \cdot z_y \cdot v_y) \cdot r_x(u_x \cdot z_y \cdot v_y)) \\ &= \sum_{x \in \pi^{-1}(y)} \text{tr}_x(r_x(t_x \cdot u_x \cdot z_y^2 \cdot v_y^2)) \end{aligned} \quad (11.44)$$

Since $r_x(v_y) = \bar{v}_y$ for all $x \in \pi^{-1}(y)$, the equality (11.38) follows from (11.42) and (11.44).

Corollary 11.8 *The trace pairing $\text{tr}_{D'} : H^0(\mathcal{O}_{D'}) \times H^0(\mathcal{O}_{D'}) \rightarrow \mathbf{Q}$ defines a symmetric non-degenerate G -invariant bilinear form, where $H^0(\mathcal{O}_{D'})_{\mathbf{Q}} = H^0(\mathcal{O}_{D'}) = \sum_{x \in D'} k(x)$. On characters of degree 0 we have an equality of discriminants*

$$\tilde{d}(H^0(\omega_{\mathcal{X}}(\mathcal{X}_S^{\text{red}} + \mathcal{J}' + \mathcal{F}')|_{D'}), \langle, \rangle) = \tilde{d}(H^0(\mathcal{O}_{D'}), \text{tr}_{D'}) = \tilde{d}(H^0(\omega_{\mathcal{X}}(\mathcal{X}_S^{\text{red}} + \mathcal{J}')|_{D'}), \langle, \rangle). \quad (11.45)$$

Proof: With the notations of Lemma 11.7, let y be a point of D . Define $F = k(y)$, and let $M_y = H^0(\mathcal{O}_{\pi^{-1}(y)})$. Then M_y is a free FG -module, and there is a natural G -invariant non-degenerate symmetric trace pairing $t_y : M_y \times M_y \rightarrow F = k(y)$. The pairing $\langle, \rangle_y^\dagger$ appearing in Lemma 11.7 is simply $t_{F/\mathbf{Q}} \circ t_y$. Let \bar{v}_y be the element of $F^* = k(y)^*$ defined in Lemma 11.7. We then have another non-degenerate pairing $t'_y : M_y \times M_y \rightarrow F$ defined by $t'_y(m, m') = t_y(c_y m, m')$ when $c_y = (\bar{v}_y)^{-1}$. This gives a non-degenerate pairing $\langle, \rangle'_y : M_y \times M_y \rightarrow \mathbf{Q}$ via $\langle, \rangle'_y = t_{F/\mathbf{Q}} \circ t'_y$. Lemma 11.7 gives an isometry between $(M_y, \langle, \rangle'_y)$ and the Hermitian module $(H^0(\omega_X(J')|_{\pi^{-1}(y)}), \langle, \rangle_y)$. This Lemma also shows that the orthogonal sum over $y \in D$ of the latter modules is isometric to $(H^0(\omega_X(J')|_{D'}), \langle, \rangle)$. Because $(H^0(\mathcal{O}_{D'}), \text{tr}_{D'})$ is the orthogonal sum over $y \in D'$ of $(M_y, \langle, \rangle'_y)$, Corollary A.3 shows

$$\tilde{d}(H^0(\omega_{\mathcal{X}}(\mathcal{X}_S^{\text{red}} + \mathcal{J}' + \mathcal{F}')|_{D'}), \langle, \rangle) = \tilde{d}(\kappa(H^0(\omega_{\mathcal{X}}(\mathcal{X}_S^{\text{red}} + \mathcal{J}' + \mathcal{F}')|_{D'})), \text{tr}_{D'}) \quad (11.46)$$

where

$$\kappa : H^0(\omega_X(J')|_{D'}) = H^0(\omega_{\mathcal{X}}(\mathcal{X}_S^{\text{red}} + \mathcal{J}' + \mathcal{F}')|_{D'})_{\mathbf{Q}} \rightarrow H^0(\mathcal{O}_{D'})$$

is the homomorphism defined in (11.35). Now, since $\pi^{-1}(\omega_{\mathcal{Y}}(\mathcal{Y}_S^{\text{red}} + \mathcal{J} + \mathcal{F})) = \omega_{\mathcal{X}}(\mathcal{X}_S^{\text{red}} + \mathcal{J}' + \mathcal{F}')$, we see from the definition of κ that $\kappa(H^0(\omega_{\mathcal{X}}(\mathcal{X}_S^{\text{red}} + \mathcal{J}' + \mathcal{F}')|_{D'})) = \mathcal{O}_{D'} \otimes_{\mathcal{O}_{\mathcal{D}}} L$ for some locally free rank one $\mathcal{O}_{\mathcal{D}}$ -module L inside $\mathcal{O}_{D'}$. Thus as $\mathcal{O}_{D'}$ - G modules, L and $\mathcal{O}_{D'}$ are the pull backs of $\mathcal{O}_{\mathcal{D}}$ -modules having isomorphic general fibers. A repeated application of Proposition 7.2 then shows

$$d(\kappa(H^0(\omega_{\mathcal{X}}(\mathcal{X}_S^{\text{red}} + \mathcal{J}' + \mathcal{F}')|_{D'})), \text{tr}_{D'}) = d(H^0(\mathcal{O}_{D'}), \text{tr}_{D'}) \cdot c \quad (11.47)$$

where c is an integral combination of classes in $HCl(\mathbf{Z}G)$ of finite $\mathbf{Z}G$ -modules of the form $\mathcal{O}_{D'} \otimes_{\mathcal{O}_{\mathcal{D}}} m$ with m a finite $\mathcal{O}_{\mathcal{D}}$ -module. However, the tameness of the action of G on D' implies all of the latter classes are integral combinations of classes of the form $(\mathbf{Z}/r)G$ with r a rational prime. The restriction of such classes to degree 0 characters is trivial, so we conclude from (11.47) that

$$\tilde{d}(\kappa(H^0(\omega_{\mathcal{X}}(\mathcal{X}_S^{\text{red}} + \mathcal{J}' + \mathcal{F}')|_{D'})), \text{tr}_{D'}) = \tilde{d}(H^0(\mathcal{O}_{D'}), \text{tr}_{D'}). \quad (11.48)$$

Combining (11.46) and (11.48) shows the first equality in (11.45), and the second equality is proved in exactly the same way.

Lemma 11.9 *In the derived category $D(G, \text{Spec}(\mathbf{Z}))$, $R_*(\omega_{\mathcal{X}}(\mathcal{X}_S^{\text{red}}))$ is isomorphic to a complex of projective $\mathbf{Z}G$ -modules*

$$[H^0(\omega_{\mathcal{X}}(\mathcal{X}_S^{\text{red}} + \mathcal{J}')) \xrightarrow{\delta_0} H^0(\omega_{\mathcal{X}}(\mathcal{X}_S^{\text{red}} + \mathcal{J}')|_{\mathcal{J}'})] \quad (11.49)$$

whose terms are in degrees 0 and 1, respectively. The homomorphism δ_0 in this complex is induced by the homomorphism τ in the complex of sheaves

$$0 \longrightarrow \omega_{\mathcal{X}}(\mathcal{X}_S^{\text{red}}) \longrightarrow \omega_{\mathcal{X}}(\mathcal{X}_S^{\text{red}} + \mathcal{J}') \xrightarrow{\tau} \omega_{\mathcal{X}}(\mathcal{X}_S^{\text{red}} + \mathcal{J}')|_{\mathcal{J}'} \longrightarrow 0. \quad (11.50)$$

Similarly, $R_*(\mathcal{O}_{\mathcal{X}})[1]$ is isomorphic to the complex of projective $\mathbf{Z}G$ -modules

$$[H^0(\mathcal{O}_{\mathcal{J}'}) \xrightarrow{\delta_{-1}} H^1(\mathcal{O}_{\mathcal{X}}(-\mathcal{J}'))] \quad (11.51)$$

whose terms are in degrees -1 and 0 , respectively. The homomorphism δ_{-1} in this complex is induced by the first boundary map in the long exact cohomology sequence associated to the exact sequence of sheaves

$$0 \rightarrow \mathcal{O}_{\mathcal{X}}(-\mathcal{J}') \rightarrow \mathcal{O}_{\mathcal{X}} \rightarrow \mathcal{O}_{\mathcal{J}'} \rightarrow 0. \quad (11.52)$$

Proof: The cohomology of $\omega_{\mathcal{X}}(\mathcal{X}_S^{\text{red}})$ is trivial outside dimensions 0 and 1. Since the terms of (11.49) are projective, we find from the cohomology of (11.50) that there is a quasi-isomorphism between (11.49) and $R_*(\omega_{\mathcal{X}}(\mathcal{X}_S^{\text{red}}))$. The assertion about (11.51) is proved similarly.

Corollary 11.10 *The hypercohomology $R_*(\omega_{\mathcal{X},S}^{\bullet}(0)[1])$ appearing in Theorem 6.4 is quasi-isomorphic to the direct sum*

$$[H^0(\mathcal{O}_{\mathcal{J}'}) \rightarrow H^1(\mathcal{O}_{\mathcal{X}}(-\mathcal{J}')) \bigoplus H^0(\omega_{\mathcal{X}}(\mathcal{X}_S^{\text{red}} + \mathcal{J}')) \rightarrow H^0(\omega_{\mathcal{X}}(\mathcal{X}_S^{\text{red}} + \mathcal{J}')|_{\mathcal{J}'})] \quad (11.53)$$

of the complexes in (11.49) and (11.51), whose terms are in dimensions -1 , 0 and 1 . The perfect pairing \langle, \rangle on $R_*(\omega_{\mathcal{X},S}^{\bullet}(0)[1])_{\mathbf{Q}}$ in Theorem 6.4 is induced by the pairing \langle, \rangle_0 on

$$H^1(\mathcal{O}_{\mathcal{X}}(-\mathcal{J}')) \bigoplus H^0(\omega_{\mathcal{X}}(\mathcal{J}')) \quad (11.54)$$

together with the pairing \langle, \rangle_J on

$$H^0(\mathcal{O}_{\mathcal{J}'}) \bigoplus H^0(\omega_{\mathcal{X}}(\mathcal{J}')|_{\mathcal{J}'}) \quad (11.55)$$

which associated to the adjunction isomorphism $\omega_{\mathcal{X}}(\mathcal{J}')|_{\mathcal{J}'} \rightarrow \omega_{\mathcal{J}'}$ and the trace homomorphism $H^0(\omega_{\mathcal{J}'}) \rightarrow \mathbf{Q}$.

Proof: In view of Lemma 11.9, we have only to show the assertions made about \langle, \rangle . Note that (11.54) is the tensor product of the middle term of (11.53) with \mathbf{Q} over \mathbf{Z} , while (11.55) is the tensor product of the sum of the degree -1 and degree 1 terms of (11.53) with \mathbf{Q} over \mathbf{Z} . The cohomology pairings defining \langle, \rangle in (11.2) were specified using the duality pairings on the cohomology of $\mathcal{O}_{\mathcal{X}}$ and $\omega_{\mathcal{X}} = \Omega_{\mathcal{X}/\mathbf{Q}}^1$. Thus the fact that these pairings result in the way described in Corollary 11.10 follows from the fact that duality isomorphisms are defined in the derived category and are compatible with adjunction. Alternately, one could give a more explicit proof along the lines of the proof of Proposition 11.5.

Corollary 11.11 *Let $(Z_J, \langle, \rangle_J)$ be the Hermitian module*

$$Z_J = H^0(\mathcal{O}_{\mathcal{J}'}) \oplus H^0(\omega_{\mathcal{X}}(\mathcal{X}_S^{red} + \mathcal{J}')|_{\mathcal{J}'})$$

with the pairing \langle, \rangle_J specified in Corollary 11.10. One has

$$\chi_H(\mathcal{X}, G)(0) = d(R, (\omega_{\mathcal{X}, S}^\bullet(0)[1]), \langle, \rangle) \quad (11.56)$$

$$= d(M_0(\mathcal{J}'), \langle, \rangle_0) \cdot d(Z_J, \langle, \rangle_J)^{-1} \quad (11.57)$$

with the notations of Corollary 11.6. Hence (11.56), (11.33) and (11.34) imply that on characters of degree 0, one has

$$\tilde{\chi}_H(\mathcal{X}, G)(0) = \tilde{d}(R, (\omega_{\mathcal{X}, S}^\bullet(\mathcal{J}') [1]), \langle, \rangle) \cdot \tilde{d}(Z_J, \langle, \rangle_J)^{-1}. \quad (11.58)$$

Proof: The equality in (11.56) is the definition of $\chi_H(\mathcal{X}, G)(0)$. The equality (11.57) follows from Corollary 11.10 and the definition of the discriminant of a perfect Hermitian complex; recall that $M_0(\mathcal{J}') = H^1(\mathcal{O}_{\mathcal{X}}(-\mathcal{J}')) \oplus H^0(\omega_{\mathcal{X}}(\mathcal{X}_S^{red} + \mathcal{J}'))$. The implication (11.58) is now clear.

Lemma 11.12 *Let $tr_{\mathcal{J}'} : H^0(\mathcal{O}_{\mathcal{J}'}) \times H^0(\mathcal{O}_{\mathcal{J}'}) \rightarrow \mathbf{Q}$ be the natural trace pairing. One has*

$$\tilde{d}(Z_J, \langle, \rangle_J) = \tilde{d}(H^0(\mathcal{O}_{\mathcal{J}'}), tr_{\mathcal{J}'})^2.$$

Proof: Since J is a sum of reduced closed points on the general fiber Y of \mathcal{Y} , there is an isomorphism $\omega_J \rightarrow \mathcal{O}_J$ which identifies the trace homomorphism $H^0(\omega_J) \rightarrow \mathbf{Q}$ with the trace $tr_J : H^0(\mathcal{O}_J) \rightarrow \mathbf{Q}$. This isomorphism induces by tensor product with $\mathcal{O}_{\mathcal{J}'}$ an isomorphism $\mu_{\mathcal{J}'} : \omega_{\mathcal{J}'} \rightarrow \mathcal{O}_{\mathcal{J}'}$ which identifies the trace $H^0(\omega_{\mathcal{J}'}) \rightarrow \mathbf{Q}$ with $tr_{\mathcal{J}'} : H^0(\mathcal{O}_{\mathcal{J}'}) \rightarrow \mathbf{Q}$. Define \langle, \rangle'_J to be the pairing on $H^0(\mathcal{O}_{\mathcal{J}'}) \oplus H^0(\mathcal{O}_{\mathcal{J}'})$ for which

$$\langle a \oplus b, a' \oplus b' \rangle'_J = tr_{\mathcal{J}'}(a, b') + tr_{\mathcal{J}'}(a', b). \quad (11.59)$$

From the definition of \langle, \rangle_J we see that

$$d(Z_J, \langle, \rangle_J) = d(H^0(\mathcal{O}_{\mathcal{J}'}) \oplus \mu_{\mathcal{J}'}(H^0(\omega_{\mathcal{X}}(\mathcal{X}_S^{red} + \mathcal{J}')|_{\mathcal{J}'})), \langle, \rangle'_J).$$

Since $\omega_{\mathcal{X}}(\mathcal{X}_S^{red} + \mathcal{J}')$ is the pull back of a line bundle on \mathcal{Y} , we find, as in the proof of (11.47), that

$$\tilde{d}(H^0(\mathcal{O}_{\mathcal{J}'}) \oplus \mu_{\mathcal{J}'}(H^0(\omega_{\mathcal{X}}(\mathcal{X}_S^{red} + \mathcal{J}')|_{\mathcal{J}'})), \langle, \rangle'_J) = \tilde{d}(H^0(\mathcal{O}_{\mathcal{J}'}) \oplus H^0(\mathcal{O}_{\mathcal{J}'}), \langle, \rangle'_J). \quad (11.60)$$

By Lemma B.2, one has

$$\tilde{d}(H^0(\mathcal{O}_{\mathcal{J}'}) \oplus H^0(\mathcal{O}_{\mathcal{J}'}), \langle, \rangle'_J) = \tilde{d}(H^0(\mathcal{O}_{\mathcal{J}'}), tr_{\mathcal{J}'})^2 \quad (11.61)$$

so Lemma 11.12 follows on combining (11.59), (11.60) and (11.61).

Corollary 11.13

$$\tilde{\chi}_H(\mathcal{X}, G)(0) = \tilde{d}(H^0(\mathcal{O}_{\mathcal{D}'}), tr_{\mathcal{D}'}) \cdot \tilde{d}(H^0(\mathcal{O}_{\mathcal{J}'}), tr_{\mathcal{J}'})^{-2}.$$

Proof: Combine Corollaries 11.11, 11.6 and 11.8 with Lemma 11.12.

12 Proof of Theorem 6.4

Let \mathcal{D}'^\sharp (resp. \mathcal{J}'^\sharp) be the normalization of \mathcal{D}' (resp. \mathcal{J}'). By the construction of \mathcal{D} and \mathcal{J} in Lemma 11.1, the natural morphisms $\nu_{\mathcal{D}} : \mathcal{D}'^\sharp \rightarrow \mathcal{D}'$ and $\nu_{\mathcal{J}} : \mathcal{J}'^\sharp \rightarrow \mathcal{J}'$ are isomorphisms over a non-empty open neighborhood of the branch locus of the morphisms $\mathcal{D}' \rightarrow \mathcal{D}$ and $\mathcal{J}' \rightarrow \mathcal{J}$ induced by $\pi : \mathcal{X} \rightarrow \mathcal{Y}$. Therefore the action of G on \mathcal{D}'^\sharp and \mathcal{J}'^\sharp is tame, and the quotient \mathcal{D}'^\sharp/G (resp. \mathcal{J}'^\sharp/G) is the normalization of \mathcal{D} (resp. \mathcal{J}). Furthermore, the $\mathcal{O}_{\mathcal{D}'}$ -module $\nu_{\mathcal{D},*}(\mathcal{O}_{\mathcal{D}'^\sharp})/\mathcal{O}_{\mathcal{D}'}$ is the pullback of a finite $\mathcal{O}_{\mathcal{D}}$ -module. It follows that the Euler characteristic of $\nu_{\mathcal{D},*}(\mathcal{O}_{\mathcal{D}'^\sharp})/\mathcal{O}_{\mathcal{D}'}$ in the Grothendieck group of all finite $\mathbf{Z}G$ -modules (c.f. §7) is an integral combination of classes of the form $\mathbf{F}_p G$ as p ranges over rational primes. Since \mathcal{D}'^\sharp has general fiber D' , we now find from Proposition 7.2 that

$$\tilde{d}(H^0(\mathcal{O}_{\mathcal{D}'^\sharp}), h_{D'}) = \tilde{d}(H^0(\mathcal{O}_{\mathcal{D}'}), h_{D'}).$$

Similarly,

$$\tilde{d}(H^0(\mathcal{O}_{\mathcal{J}'^\sharp}), h_{J'}) = \tilde{d}(H^0(\mathcal{O}_{\mathcal{J}'}), h_{J'}).$$

The scheme \mathcal{D}'^\sharp is the disjoint union of the spectra of rings of integers on which G acts tamely, and $h_{D'}$ corresponds to the orthogonal sum of the trace forms on these rings of integers. We now describe the main result of Cassou-Noguès and Taylor in [2] concerning the image of $d(H^0(\mathcal{O}_{\mathcal{D}'^\sharp}, h_{D'}))$ in $Ad\ HCl(\mathbf{Z}G)$.

Let $\epsilon_\infty^s(\mathcal{D}^\sharp)$ be the element of $Hom(R_G^s, J(\overline{\mathbf{Q}}))$ such that $\epsilon_\infty^s(\mathcal{D}^\sharp)(\chi_V)$ for $\chi_V \in R_G^s$ have trivial components at all finite places, and component at the archimedean place v_∞ of \mathbf{Q} given by the archimedean local constant $\epsilon_{v_\infty}(\mathcal{D}^\sharp, V)$. By [4, Cor. 5.2.3], $\epsilon_{v_\infty}^s(\mathcal{D}^\sharp, V) = \pm 1$ for $\chi_V \in R_G^s$, and $\epsilon_\infty^s(\mathcal{D}^\sharp)$ is $\Omega_{\mathbf{Q}}$ -equivariant. Thus we may view $\epsilon_\infty^s(\mathcal{D}^\sharp)$ as defining a class in $Ad\ HCl(\mathbf{Z}G)$. By restricting the main result of [2] concerning $d(H^0(\mathcal{O}_{\mathcal{D}'^\sharp}, h_{D'}))$ to characters of degree 0, we have an equality

$$\tilde{d}(H^0(\mathcal{O}_{\mathcal{D}'^\sharp}), h_{D'}) = \tilde{\epsilon}_0^s(\mathcal{D}^\sharp) \cdot \tilde{\epsilon}_\infty^s(\mathcal{D}^\sharp) \quad \text{in } Ad\ HCl(\mathbf{Z}G).$$

In deducing this from [2], it is important to use the fact proved in [13, Theorem IV.29(i)] that the non-ramified characteristic associated to symplectic representation of Galois extension of local fields is equal to 1. This allows one to compare $\tilde{\epsilon}_0^s(\mathcal{D}^\sharp)$ to the symplectic root numbers appearing in [2].

In a similar way, one has

$$\tilde{d}(H^0(\mathcal{O}_{\mathcal{J}'^\sharp}), h_{J'}) = \tilde{\epsilon}_0^s(\mathcal{J}^\sharp) \cdot \tilde{\epsilon}_\infty^s(\mathcal{J}^\sharp) \quad \text{in } Ad\ HCl(\mathbf{Z}G).$$

Thus Corollary 11.13 shows

Corollary 12.1

$$\tilde{\chi}_H(\mathcal{X}, G)(0) = \tilde{\epsilon}_0^s(\mathcal{D}^\sharp) \cdot \tilde{\epsilon}_0^s(\mathcal{J}^\sharp)^{-2} \cdot \tilde{\epsilon}_\infty^s(\mathcal{D}^\sharp) \cdot \tilde{\epsilon}_\infty^s(\mathcal{J}^\sharp)^{-2} \quad \text{in } Ad\ HCl(\mathbf{Z}G). \quad (12.1)$$

Recall that in Lemma 11.1, \mathcal{D} and \mathcal{J} were chosen so that

$$K_{\mathcal{Y}} + \mathcal{Y}_S^{red} + \mathcal{F} = \mathcal{D} - 2\mathcal{J}. \quad (12.2)$$

We now use the comparison of symplectic root numbers appearing in the proof of Theorem [4, Theorem 7.1.3] to conclude from (12.2) that

$$\tilde{\epsilon}_0^s(\mathcal{Y})^{-1} = \tilde{\epsilon}_0^s(\mathcal{D}^\sharp) \cdot \tilde{\epsilon}_0^s(\mathcal{J}^\sharp)^{-2}. \quad (12.3)$$

Note that the exponent -1 on the left arises from the computation of the degree of the restriction of $\mathcal{O}_{\mathcal{Y}}(K_{\mathcal{Y}} + \mathcal{Y}_S^{red})$ to the irreducible components of the finite fibers of \mathcal{Y} given in [4, §7.1] together with the formulas for ϵ -factors proved by Saito and used in the proof of [4, Thm. 7.1.3]. Part (c) of [4, Theorem 7.1.3] shows in a similar way that for all $\chi_V \in R_G^s$ of dimension 0, we have

$$\epsilon_{v_\infty,0}(\mathcal{Y}, V) = \epsilon_{v_\infty,0}(\mathcal{D}, V) \cdot \epsilon_{v_\infty,0}(\mathcal{J}, V)^{-2}. \quad (12.4)$$

However, since the general fiber of \mathcal{Y} is of odd dimension, we have from [4, Lemma 5.1.1(b)] that $\epsilon_{v_\infty,0}(\mathcal{Y}, V) = 1$ for all $\chi_V \in R_G^s$. Thus (12.4) shows

$$\tilde{\epsilon}_\infty^s(\mathcal{D}^\sharp) \cdot \tilde{\epsilon}_\infty^s(\mathcal{J}^\sharp)^{-2} = 1 \quad \text{in } Ad\ HCl(\mathbf{Z}G). \quad (12.5)$$

Combining (12.1), (12.3) and (12.5) proves Theorem 6.4 in view of Proposition 9.1(i).

13 Proof of Theorem 6.7

The complex $\lambda^{\leq 1}(\Omega_{\mathcal{X}/\mathbf{Z}}^1)[1]$ used to define $\chi_H(\mathcal{X}, G)$ is quasi-isomorphic to $(\mathcal{O}_{\mathcal{X}} \rightarrow \Omega_{\mathcal{X}/\mathbf{Z}}^1)[1]$, while $\omega_{\mathcal{X},S}^\bullet(0)[1]$ is the complex $(\mathcal{O}_{\mathcal{X}} \rightarrow \omega_{\mathcal{X}}(\mathcal{X}_S^{red}))[1]$. Here $\Omega_{\mathcal{X}/\mathbf{Z}}^1$ and $\omega_{\mathcal{X}}(\mathcal{X}_S^{red})$ are isomorphic on the general fiber of \mathcal{X} . In [6, §8.1] it was shown how to define and compute an Euler characteristic $\chi_{tor}(\Omega_{\mathcal{X}/\mathbf{Z}}^1 \rightarrow \omega_{\mathcal{X}}(\mathcal{X}_S^{red}))$ in the Grothendieck group $K_0T(\mathbf{Z}G)$ of all finite $\mathbf{Z}G$ -modules of finite projective dimension which is associated to the natural morphism $\Omega_{\mathcal{X}/\mathbf{Z}}^1 \rightarrow \omega_{\mathcal{X}}(\mathcal{X}_S^{red})$. (As in the proof of Proposition 9.1(ii) in §9, one can view this morphism as defining a two term complex $\mathcal{C}_{\mathcal{X}}$ in degrees -1 and 0 .) By applying Proposition 7.2 we now deduce that

$$\chi_H(\mathcal{X}, G) = \chi_H(\mathcal{X}, G)(0) \cdot t(\chi_{tor}(\Omega_{\mathcal{X}/\mathbf{Z}}^1 \rightarrow \omega_{\mathcal{X}}(\mathcal{X}_S^{red})))^{-1} \quad (13.1)$$

where $t : K_0T(\mathbf{Z}G) \rightarrow HCl(\mathbf{Z}G)$ is the homomorphism discussed in §7. It follows from [6, §8.1] that on taking images in $Ad\ HCl(\mathbf{Z}G)$ and then passing to characters of degree 0, we have

$$\tilde{t}(\chi_{tor}(\Omega_{\mathcal{X}/\mathbf{Z}}^1 \rightarrow \omega_{\mathcal{X}}(\mathcal{X}_S^{red})))^{-1} = \prod_p |\tilde{\epsilon}_p^s(\mathcal{Y}_p)|_p \quad (13.2)$$

where the product on the right is over all rational primes p , \mathcal{Y}_p is the fiber of \mathcal{Y} over p , and the character function $|\tilde{\epsilon}_p^s(\mathcal{Y}_p)|_p$ defined in §6 defines a class in $Ad\ HCl(\mathbf{Z}G)$. (Note that because ϵ -constants do not change on passing to the reduction of a subscheme, we have $|\tilde{\epsilon}_p^s(\mathcal{Y}_p)|_p = |\tilde{\epsilon}_p^s(\mathcal{Y}_p^{red})|_p = |\tilde{\epsilon}_p^s(\mathcal{X}_p^{red}/G)|_p$.) Combining (13.1), (13.2) and Theorem 6.4 now shows

$$\tilde{\chi}_H(\mathcal{X}, G) = \tilde{\epsilon}_{S,0}^s(\mathcal{Y})^{-1} \cdot \prod_p |\tilde{\epsilon}_p^s(\mathcal{Y}_p)|_p. \quad (13.3)$$

Let $b_p \subset \mathcal{Y}_p$ be the fiber at p of the branch locus of $\mathcal{X} \rightarrow \mathcal{Y}$, and let $U_p = \mathcal{Y}_p - b_p$. With the notations of §6, we have

$$\begin{aligned} |\tilde{\epsilon}_p^s(\mathcal{Y}_p)|_p \cdot \tilde{\epsilon}^s(\mathcal{Y}_p) &= |\tilde{\epsilon}_p^s(\mathcal{Y}_p)|_p \cdot \tilde{\epsilon}_p^s(\mathcal{Y}_p) \cdot \prod_{v \neq p} \tilde{\epsilon}_v^s(\mathcal{Y}_p) \\ &= |\tilde{\epsilon}_p^s(U_p)|_p \cdot \tilde{\epsilon}_p^s(U_p) \cdot |\tilde{\epsilon}_p^s(b_p)|_p \cdot \tilde{\epsilon}_p^s(b_p) \prod_{v \neq p} (\tilde{\epsilon}_v^s(U_p) \cdot \tilde{\epsilon}_v^s(b_p)) \end{aligned} \quad (13.4)$$

where the product is over all places v of \mathbf{Q} (including the infinite place) which are not equal to p . For each place v of \mathbf{Q} , let $\text{Det}^s(\mathbf{Z}_v G)^*$ be the restriction to symplectic characters of the determinant subgroup $\text{Det}((\mathbf{Z}_v G)^*)$. By [3, Thm. 5.12, 5.13], $|\tilde{\epsilon}_p^s(U_p)|_p \cdot \tilde{\epsilon}_p^s(U_p)$ lies in $\text{Det}^s(\mathbf{Z}_p G)^*$, while $\tilde{\epsilon}_v^s(U_p)$ lies in $\text{Det}^s((\mathbf{Z}_v G)^*)$ for $v \neq p$. Hence these terms in (13.4) have trivial class in $\text{Ad } HCl(\mathbf{Z}G)$. Thus (13.4) gives

$$|\tilde{\epsilon}_p^s(\mathcal{Y}_p)|_p = \tilde{\epsilon}^s(\mathcal{Y}_p)^{-1} \cdot |\tilde{\epsilon}_p^s(b_p)|_p \cdot \tilde{\epsilon}_p^s(b_p) \cdot \prod_{v \neq p} \tilde{\epsilon}_v^s(b_p) \quad \text{in } \text{Ad } HCl(\mathbf{Z}G). \quad (13.5)$$

From §6 we have

$$\tilde{\epsilon}^s(\mathcal{Y}) = \tilde{\epsilon}_{S,0}^s(\mathcal{Y}) \prod_p \tilde{\epsilon}^s(\mathcal{Y}_p). \quad (13.6)$$

Furthermore,

$$\tilde{\epsilon}^s(b_p) = \tilde{\epsilon}_p^s(b_p) \cdot \prod_{v \neq p} \tilde{\epsilon}_v^s(b_p) \quad (13.7)$$

by definition. Therefore combining (13.3), (13.5), (13.6) and (13.7) shows

$$\tilde{\chi}_H(\mathcal{X}, G) = \tilde{\epsilon}^s(\mathcal{Y})^{-1} \cdot \prod_p (\tilde{\epsilon}^s(b_p) \cdot |\tilde{\epsilon}_p^s(b_p)|_p) \quad \text{in } \text{Ad } HCl(\mathbf{Z}G).$$

By Proposition 9.1(ii), this implies Theorem 6.7.

A Appendix: Behavior of discriminants under base changes.

With the notations of §4 and 5, let $t_{F/\mathbf{Q}} : F \rightarrow \mathbf{Q}$ be the trace function associated to a number field F . We will also use $t_{F/\mathbf{Q}}$ for the induced trace homomorphism $FG \rightarrow \mathbf{Q}G$. We can view a locally free $O_F G$ -module M also as a locally free $\mathbf{Z}G$ -module. We fix a non-degenerate Hermitian pairing $h : M_F \times M_F \rightarrow FG$, where $M_F = F \otimes_{O_F} M$. On $M_{\mathbf{Q}} = \mathbf{Q} \otimes_{\mathbf{Z}} M$ we have the Hermitian $\mathbf{Q}G$ -module pairing $h_{\mathbf{Q}} : M_{\mathbf{Q}} \times M_{\mathbf{Q}} \rightarrow \mathbf{Q}G$ be defined by $h(m, m') = t_{F/\mathbf{Q}}(h_F(m, m'))$ for $m, m' \in M$ and then extended to all $m, m' \in M_{\mathbf{Q}}$ by \mathbf{Q} -bilinearity.

Choose a (finite) left transversal $\{\sigma\}_{\sigma}$ for Ω_F inside $\Omega_{\mathbf{Q}}$. For an arbitrary abelian $\Omega_{\mathbf{Q}}$ -module A , one has a norm map

$$\mathcal{N}_{F/\mathbf{Q}} : \text{Hom}_{\Omega_F}(R_G, A) \rightarrow \text{Hom}_{\Omega_{\mathbf{Q}}}(R_G, A) \quad (A.1)$$

defined by

$$\mathcal{N}_{F/\mathbf{Q}}(f)(\chi) = \prod_{\sigma} \sigma(f(\sigma^{-1}(\chi)))$$

for $f \in \text{Hom}_{\Omega_F}(R_G, A)$ and $\chi \in R_G^s$. Recall that $\bar{a} = \sum_{g \in G} a_g g^{-1}$ if $a = \sum_{g \in G} a_g g \in FG$. The following result is proved in [13, Thm. IV.11].

Lemma A.1 (Fröhlich) *The pairing $h_{\mathbf{Q}}$ is non-degenerate. The map $\mathcal{N}_{F/\mathbf{Q}}$ in (A.1) induces a homomorphism $\Lambda_{F/\mathbf{Q}} : HCl(O_F G) \rightarrow HCl(\mathbf{Z}G)$. We have*

$$\Lambda_{F/\mathbf{Q}}(d(M, h_F)) = d(M, h_{\mathbf{Q}}) \cdot d(O_F G, \beta)^{-r}$$

where $r = \text{rank}_{O_F G}(M)$ and $\beta : FG \times FG \rightarrow \mathbf{Q}G$ is the Hermitian pairing defined by $\beta(a, a') = t_{F/\mathbf{Q}}(\bar{a} \cdot a')$ for $a, a' \in FG$.

Lemma A.2 *Suppose $c \in F^*$, and that we define a new Hermitian pairing $h'_F : M_F \times M_F \rightarrow FG$ by*

$$h'_F(m, m') = c \cdot h_F(m, m') \quad \text{for } m, m' \in M_F.$$

The dimension $\dim(\chi)$ of each $\chi \in R_G^s$ is even. Define $f(c) \in \text{Hom}_{\Omega_F}(R_G^s, \overline{\mathbf{Q}}^*)$ by $f(c)(\chi) = c^{\dim(\chi)/2}$ for $\chi \in R_G^s$. Then

$$Pf((h'_F(x_i, x_j))_{i,j}) = f(c) \cdot Pf((h_F(x_i, x_j))_{i,j}) \quad (\text{A.2})$$

in $\text{Hom}_{\Omega_F}(R_G^s, \overline{\mathbf{Q}}^*)$.

Proof: Since $\dim(\chi)$ is even if $\chi \in R_G^s$, $f(c)$ is well defined. Clearly

$$(h'_F(v_i, v_j))_{i,j} = c \cdot (h_F(v_i, v_j))_{i,j}.$$

Since Pfaffians do not change on extending h_F from F to $\overline{\mathbf{Q}}$, we find from [13, Prop. II.4.5] that

$$Pf((h'_F(x_i, x_j))_{i,j})(\chi) = \text{Det}_{\chi}(\sqrt{c} \cdot I_n) \cdot Pf((h_F(x_i, x_j))_{i,j})(\chi)$$

for each symplectic character $\chi \in R_G^s$, where \sqrt{c} is a any choice of square root of c in $\overline{\mathbf{Q}}$. Since $\dim(\chi)$ is even, this implies (A.2).

Corollary A.3 *Let $c, f(c), h_F$ and h'_F be as in Lemma A.2. As in Lemma A.1, let $h_{\mathbf{Q}} = t_{F/\mathbf{Q}} \circ h_F$ and $h'_{\mathbf{Q}} = t_{F/\mathbf{Q}} \circ h'_F$ be the induced Hermitian forms on $M_{\mathbf{Q}}$. The function $\mathcal{N}(f(c)) \in \text{Hom}_{\Omega_{\mathbf{Q}}}(R_G^s, \overline{\mathbf{Q}}^*)$ then sends $\chi \in R_G^s$ to*

$$\mathcal{N}(f(c))(\chi) = \text{Norm}_{F/\mathbf{Q}}(c)^{\dim(\chi)/2}. \quad (\text{A.3})$$

Let $[1 \times \mathcal{N}(f(c))]$ be the image of

$$1 \times \mathcal{N}(f(c)) \in \frac{\text{Det}(J(\mathbf{Q}G))}{\text{Det}(U(\mathbf{Z}G))} \times \text{Hom}_{\Omega_F}(R_G^s, \overline{\mathbf{Q}}^*)$$

in $HCl(\mathbf{Z}G)$. Then

$$d(M, h'_{\mathbf{Q}}) = [1 \times \mathcal{N}(f(c))] \cdot d(M, h_{\mathbf{Q}}). \quad (\text{A.4})$$

Proof: Since h'_F and h_F are pairings on the same FG -module, Lemmas A.1 and A.2 show that (A.4) holds. The only thing we need to check is that (A.3) holds. However, by definition

$$\begin{aligned} \mathcal{N}(f(c))(\chi) &= \prod_{\sigma} \sigma(f(\sigma^{-1}(\chi))) \\ &= \prod_{\sigma} \sigma(c^{\dim(\sigma^{-1}(\chi))/2}) \\ &= \prod_{\sigma} \sigma(c)^{\dim(\chi)/2} \end{aligned} \tag{A.5}$$

$$= \text{Norm}_{F/\mathbf{Q}}(c)^{\dim(\chi)/2} \tag{A.6}$$

since σ is a transversal for Ω_F in $\Omega_{\mathbf{Q}}$.

Corollary A.4 *With the notations of Corollary A.3, the degree 0 classes $\tilde{d}(M, h_{\mathbf{Q}})$ and $\tilde{d}(M, h'_{\mathbf{Q}})$ are equal.*

Lemma A.5 *Suppose F' is a finite extension of F with ring of integers $O_{F'}$. Define $M' = M \otimes_{O_F} O_{F'}$. One has a natural isomorphism $M'_{F'} = M_F \otimes_F F'$, The pairing $h_F : M_F \times M_F \rightarrow FG$ induces a Hermitian pairing $h_{F'} : M'_{F'} \times M'_{F'} \rightarrow F'G$ via this isomorphism. Let $h''_{\mathbf{Q}}$ be the pairing $t_{F'/\mathbf{Q}} \circ h_{F'}$ on $M'_{\mathbf{Q}} = M' \otimes_{\mathbf{Z}} \mathbf{Q}$. Let $n = \text{rank}_{FG}(M)$, and let $\delta = d_{F'/\mathbf{Q}} \cdot d_{F/\mathbf{Q}}^{-[F':F]} \in \mathbf{Q}$. Define*

$$z = 1 \times z_s \in \frac{\text{Det}(J(\mathbf{Q}G))}{\text{Det}(U(\mathbf{Z}G))} \times \text{Hom}_{\Omega_F}(R_G^s, \overline{\mathbf{Q}}^*)$$

by letting $z_s(\chi)$ equal $\delta^{n \cdot \dim(\chi)/2}$ for all $\chi \in R_G^s$. Let $[z]$ be the image of z in $HCl(\mathbf{Z}G)$. Then

$$d(M', h''_{\mathbf{Q}}) = d(M, h_{\mathbf{Q}})^{[F':F]} \cdot [z]. \tag{A.7}$$

Proof: If A is an Ω_F -module, the relative norm operator $\mathcal{N}_{F'/F} : \text{Hom}_{\Omega_{F'}}(R_G, A) \rightarrow \text{Hom}_{\Omega_F}(R_G, A)$ acts as multiplication by $[F' : F]$ on the subgroup $\text{Hom}_{\Omega_F}(R_G, A)$ of $\text{Hom}_{\Omega_{F'}}(R_G, A)$. Hence combining the going-up result of [13, Theorem IV.9] with the going down result in [13, Theorem IV.11] shows

$$d(M', t_{F'/F} \circ h_{F'}) = d(M, h_F)^{[F':F]} \cdot d(O_{F'}G, \beta_{F'/F})^n. \tag{A.8}$$

in $HCl(O_FG)$, where $\beta_{F'/F} : F'G \times F'G \rightarrow FG$ is the Hermitian form resulting from the trace $t_{F'/F}$. Another application of the going down result shows

$$d(M', h''_{\mathbf{Q}}) = \mathcal{N}_{F/\mathbf{Q}}(d(M', t_{F'/F} \circ h_{F'})) \cdot d(O_FG, \beta_{F/\mathbf{Q}})^{n \cdot [F':F]} \tag{A.9}$$

and

$$d(M, h_{\mathbf{Q}}) = \mathcal{N}_{F/\mathbf{Q}}(d(M, h_F)) \cdot d(O_FG, \beta_{F/\mathbf{Q}})^n \tag{A.10}$$

where $\mathcal{N} : HCl(O_FG) \rightarrow HCl(\mathbf{Z}G)$ is the norm map. Combining (A.8), (A.9) and (A.10) gives

$$d(M', h''_{\mathbf{Q}})/d(M, h_{\mathbf{Q}})^{[F':F]} = \mathcal{N}_{F/\mathbf{Q}}(d(O_{F'}G, \beta_{F'/F})^n). \tag{A.11}$$

The right hand side of (A.11) is the image of $\mathcal{N}_{F/\mathbf{Q}}(d(O_{F'}, t_{F'/F})^n)$ in $HCl(\mathbf{Z}G)$ under the going up homomorphism $HCl(\mathbf{Z}) \rightarrow HCl(\mathbf{Z}G)$ induced by the ring inclusion of \mathbf{Z} into $\mathbf{Z}G$. In $HCl(\mathbf{Z})$ we have the usual relative discriminant formula

$$d(O_{F'}, t_{F'/\mathbf{Q}}) = \mathcal{N}_{F/\mathbf{Q}}(d(O_{F'}, t_{F'/F})) \cdot d(O_F, t_{F/\mathbf{Q}})^{[F':F]}. \quad (\text{A.12})$$

Thus 5.2 shows that $\mathcal{N}_{F/\mathbf{Q}}(d(O_{F'}, t_{F'/F}))$ is identified with δ under the isomorphism $HCl(\mathbf{Z}) = \mathbf{Q}^*$ in (5.2). By [13, Theorem IV.11], the going up homomorphism $HCl(\mathbf{Z}) \rightarrow HCl(\mathbf{Z}G)$ above is induced by the homomorphism $R_G \rightarrow K_e = \mathbf{Z}$ which sends a character to its dimension. Hence the right hand side of (A.11) is equal to $[z]$, and (A.7) is proved.

Corollary A.6 *With the notations of Lemma A.5, the degree 0 classes $\tilde{d}(M', h_{\mathbf{Q}}'')$ and $\tilde{d}(M, h_{\mathbf{Q}})^{[F':F]}$ are equal.*

B Appendix: Exact sequences of Hermitian modules.

With the notations of Appendix A, let $\langle, \rangle : M_F \times M_F \rightarrow F$ be a symmetric G -equivariant F -bilinear non-degenerate form. By Lemma 2.1, such \langle, \rangle correspond bijectively to non-degenerate Hermitian forms $\langle, \rangle_G : M_F \times M_F \rightarrow FG$ via the usual formula

$$\langle m, m' \rangle_G = \sum_{g \in G} \langle m, gm' \rangle g^{-1}.$$

Lemma B.1 *Suppose $0 \rightarrow W \xrightarrow{\alpha} M \xrightarrow{\beta} T \rightarrow 0$ is an exact sequence of locally free $O_F G$ -modules. Suppose also that there is a G -equivariant perfect bilinear pairing $\langle, \rangle' : W_F \times T_F \rightarrow F$ such that*

$$\langle \alpha(w), m \rangle = \langle w, \beta(m) \rangle' \quad \text{for } w \in W_F, m \in M_F.$$

Then $(M_F, \langle, \rangle_G)$ is a hyperbolic space in the sense of [13, p. 164, 52]. We have an equality of discriminants

$$d(M, \langle, \rangle_G) = d(W \oplus T, \langle, \rangle_G'') \quad (\text{B.1})$$

where $\langle, \rangle'' : (W \oplus T)_F \times (W \oplus T)_F \rightarrow F$ is the symmetric G -equivariant bilinear form defined by

$$\langle w \oplus t, w' \oplus t' \rangle'' = \langle w, t' \rangle + \langle w', t \rangle.$$

Proof: Since T_F is a free FG -module, we can find a G -equivariant section $\lambda : T_F \rightarrow M_F$ of $\beta \otimes_{O_F} F$. Fix $t \in T_F$. Since λ is F -linear and \langle, \rangle is F -bilinear, the function $f_t : T_F \rightarrow F$ defined by

$$f_t(t') = -\langle \lambda(t), \lambda(t') \rangle / 2$$

for $t' \in T_F$ lies in $\text{Hom}_F(T_F, F)$. Because \langle, \rangle' is perfect and G -equivariant, it defines a G -isomorphism between W_F and $\text{Hom}_F(T_F, F)$. Hence there is a unique $\tau(t) \in W_F$ such that

$$-\langle \lambda(t), \lambda(t') \rangle / 2 = \langle \tau(t), t' \rangle' \quad \text{for } t' \in T_F.$$

A straightforward computation shows that the function $\tau : T_F \rightarrow W_F$ sending t to $\tau(t)$ lies in $\text{Hom}_{FG}(T_F, W_F)$.

We can now define a new section $\lambda' : T_F \rightarrow M_F$ by $\lambda' = \lambda + (\alpha \circ \tau)$. Let $\eta : W_F \oplus T_F \rightarrow M_F$ be the $\mathbf{Q}G$ -isomorphism defined by $\eta(w \oplus t) = \alpha(w) + \lambda'(t)$. Another easy calculation shows η defines an isometry between the hyperbolic pairing \langle, \rangle_G'' on $(W \oplus T)_F$ and $(M_F, \langle, \rangle_G)$.

It remains to prove the equality of discriminants (B.1). Following the recipe in Definition 4.4, we choose the basis $\{v_i\}_i$ for M_F over FG so that a subset $\{v_j\}_j$ of this basis is a basis for W_F over FG , the complement $\{v'_l\}_l$ of this subset in $\{v_i\}_i$ projects to a basis for T_F over FG , and the matrix $\langle v_i, v'_l \rangle_{i,l}$ is a block matrix with diagonal blocks having all entries equal to 0. We can choose local bases of M in such a way that the transition matrix between $\{v_i\}_i$ and each local basis is a block upper triangular matrix, with the diagonal blocks coming from transition matrices associated to T and W . The equality $d(T, \langle, \rangle_G) = d(M \oplus W, \langle, \rangle_G'')$ now follows from Definition 4.4, since our choice of global bases makes the appropriate Pfaffian character functions equal for these two discriminants, and the determinant of a block upper triangular matrix is the product of the determinants of the diagonal blocks.

We will later need to consider the following special case:

Lemma B.2 *Suppose that $W = T$ in Lemma B.1, so that \langle, \rangle' defines a symmetric G -equivariant non-degenerate bilinear form on W_F . Then*

$$\tilde{d}(M, \langle, \rangle_G) = \tilde{d}(W \oplus T, \langle, \rangle_G'') = \tilde{d}(W, \langle, \rangle_G')^2. \quad (\text{B.2})$$

Proof: Choose local and global bases for W as a locally free $O_F G$ -module, and then use copies of these bases for T . Applying the recipe for computing Pfaffians, we see that the local transition matrices between the resulting global and the local bases for $W \oplus T$ are block diagonal, with two diagonal blocks both equal to the corresponding matrices for W and two off diagonal blocks equal to 0. The determinant of such a matrix is the square of the determinant of one of the diagonal blocks. The Pfaffian matrix for $W \oplus T$ has, on the other hand, two diagonal blocks equal to 0, and two off diagonal blocks each equal to the (symmetric) Pfaffian matrix for W . On applying to this block matrix the matrix representation associated to a symplectic character χ , we see that we must compute the Pfaffian $Pf(B)$ of a 2×2 block matrix B having diagonal blocks equal to 0 and off diagonal blocks each equal to a matrix A which is symmetric with respect to the appropriate involution j . Thus as in [13, Prop. II.3.1], $A = P \cdot P^j$ for some matrix P , with $Pf(A) = \det(P)$. To compute $Pf(B)$, we use the trick of [13, p. 43]. Thus

$$B = \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix} = \begin{pmatrix} 0 & PP^j \\ PP^j & 0 \end{pmatrix} = R \cdot R^j$$

when

$$R = \sqrt{2}^{-1} \begin{pmatrix} P & -\sqrt{-1}P \\ P & \sqrt{-1}P \end{pmatrix}$$

Thus

$$Pf(B) = \det(R) = \sqrt{-1}^{\dim(\chi)} \det(P)^2 = (-1)^{\dim(\chi)/2} \det(P)^2 = (-1)^{\dim(\chi)/2} Pf(A)^2$$

where $\dim(\chi)$ is even since χ is symplectic. Taking into account what we have already shown concerning determinants of transition matrices at finite places, we get (B.2) on restriction to characters of degree 0.

C Appendix: Computing Pfaffians

The object of this section is to describe another way of computing the Pfaffian function $Pf((h(v_i, v_j))_{i,j})$ appearing in Definition 4.4. This will be needed to complete the proof of Theorem 2.5 in Appendix D.

As before, let M_F be a free FG -module, and let $h : M_F \times M_F \rightarrow FG$ be a non-degenerate Hermitian pairing. There is a unique non-degenerate G -invariant bilinear form $h_1 : M_F \times M_F \rightarrow F$ such that

$$h(m, m') = \sum_{g \in G} h_1(m, gm') g^{-1}$$

for all $m, m' \in M_F$.

Let V be a symplectic representation of G over \overline{F} , with alternating non-degenerate G -invariant bilinear form $h_2 : V \times V \rightarrow \overline{F}$. By ([16, §XIV.9]) $\dim_{\overline{F}}(V) = 2n$ is even, and V has a symplectic basis $\{w_i\}_{i=1}^{2n}$ relative to the form h_2 such that $h_2(w_i, w_{i'}) = 0$ unless $\{i, i'\} = \{2j-1, 2j\}$ for some integer j , and $h_2(w_{2j-1}, w_{2j}) = -h_2(w_{2j}, w_{2j-1}) = 1$. Let $T : G \rightarrow Mat_{2n}(\overline{F})$ be the representation of G on V relative to the basis $\{w_i\}_i$, and extend T as before to an algebra homomorphism $T : FG \rightarrow Mat_{2n}(\overline{\mathbf{Q}})$.

Let $\{y_i\}_{i=1}^q$ be a basis for $(\overline{\mathbf{Q}})^q$. We fix the symplectic basis $\{y_i \otimes w_j\}_{i,j}$ for the orthogonal sum $V^{\perp q} = (\overline{\mathbf{Q}})^q \otimes_{\overline{\mathbf{Q}}} V$ of q copies of V . Let $H = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and let $\mathcal{H} = \text{diag}(H, \dots, H)$ be the block matrix in $Mat_{2nq}(\overline{\mathbf{Q}})$ having nq diagonal blocks equal to H and all other 2×2 blocks equal to 0. Then \mathcal{H} is the matrix of the orthogonal sum $h_2^{\perp q}$ of q copies of h_2 relative to the basis $\{y_i \otimes w_j\}_{i,j}$ of $V^q = \overline{\mathbf{Q}}^{2nq}$, in the sense that

$$h_2^{\perp q}(x, y) = x^{tr} \mathcal{H} y$$

if x and y are column vectors representing elements of V^q relative to this basis, where x^{tr} is the transpose of x . One has $\mathcal{H}^{tr} = -\mathcal{H} = \mathcal{H}^{-1}$. The symplectic involution J associated to $h_2^{\perp q}$ sends $P \in Mat_{2nq}(\overline{\mathbf{Q}})$ to $P^J = \mathcal{H}^{-1} P^{tr} \mathcal{H}$, since

$$h_2^{\perp q}(P^J x, y) = h_2^{\perp q}(\mathcal{H}^{-1} P^{tr} \mathcal{H} x, y) = x^{tr} \mathcal{H}^{tr} P (\mathcal{H}^{-1})^{tr} \mathcal{H} y = x^{tr} \mathcal{H} P y = h_2^{\perp q}(x, Py).$$

Let $\{v_i\}_{i=1}^q$ be a basis over FG for M_F . Then as in the §4, $A = \{h(v_i, v_j)\}_{i,j=1}^q$ is a matrix in $Mat_q(FG)$, and $T(A) = \{T(h(v_i, v_j))\}_{i,j=1}^q$ is a matrix in $Mat_{2nq}(\overline{\mathbf{Q}})$. Here $T(A)$ is the matrix of the action of A on the orthogonal sum $V^{\perp q}$. Since h is a Hermitian pairing, we find, as in §4, that $T(A)$ is symmetric with respect to the involution J defined above for $h_2^{\perp q}$. Thus

$$T(A)^J = \mathcal{H}^{-1} T(A)^{tr} \mathcal{H} = T(A).$$

By definition, if χ is the character of the representation V , then

$$Pf((h(v_i, v_j))_{i,j})(\chi) = Pf^J T(A).$$

Our goal is to compute $Pf^J T(A)$ using an alternating bilinear form on $(M \otimes_F V)^G$.

Lemma C.1 *There is an unique isomorphism of $\overline{\mathbf{Q}}$ -vector spaces $\psi : V^q \rightarrow (M \otimes_F V)^G$ such that $\psi(y_i \otimes w_j) = Tr_G(v_i \otimes w_j)$ for all i, j , where $Tr_G = \sum_{g \in G} g$ is the trace operator.*

Proof: Let $(M \otimes_F V)_G$ be the G -covariants of $M \otimes_F V$. Since V is a representation of G over a field of characteristic 0, the trace homomorphism $Tr_G : (M \otimes_F V)_G \rightarrow (M \otimes_F V)^G$ is an isomorphism. Because $M = \bigoplus_{i=1}^q FGv_i$ is free on the v_i , the images of the elements of $\{v_i \otimes w_j\}_{i,j}$ in $(M \otimes_F V)_G$ are distinct and form a basis of $(M \otimes_F V)_G$, and the Lemma is now clear.

Definition C.2 *Let $h_3 : (M \otimes_F V) \times (M \otimes_F V) \rightarrow \overline{F}$ be the unique alternating bilinear form defined by*

$$h_3(m \otimes v, m' \otimes v') = h_1(m, m') \cdot h_2(v, v')$$

for $m, m' \in M$ and $v, v' \in V$. Let h_3^G be the restriction of h_3 to $(M \otimes_F V)^G \times (M \otimes_F V)^G$.

Proposition C.3 *Let B be the matrix $(h_3^G(Tr_G(v_i \otimes w_j), Tr_G(v_{i'} \otimes w_{j'})))_{i,i',j,j'}$ in $Mat_{2nq}(\overline{\mathbf{Q}})$, where i and i' range over $\{1, \dots, q\}$ and j and j' range over $\{1, \dots, 2n\}$. Then $B^{tr} = -B$, and*

$$-\#G \cdot \mathcal{H} \cdot T(A) = B. \quad (\text{C.1})$$

Proof: Fix i and j , and consider the block of $T(A)$ of size $2n \times 2n$ given by

$$T(h(v_i, v_j)) = \sum_{g \in G} h_1(x_i, gx_j) T(g^{-1}).$$

For $1 \leq k, l \leq 2n$, the entry at position k, l in this block is

$$T(h(v_i, v_j))_{k,l} = \sum_{g \in G} h_1(x_i, gx_j) T(g^{-1})_{k,l} \quad (\text{C.2})$$

where

$$g^{-1}w_k = T(g^{-1})w_k = \sum_{u=1}^{2n} T(g^{-1})_{u,k} w_u. \quad (\text{C.3})$$

We now compute

$$\begin{aligned} h_3^G(Tr_G(v_i \otimes w_k), Tr_G(v_j \otimes w_l)) &= \sum_{\tau, \sigma \in G} h_3(\tau v_i \otimes \tau w_k, \sigma v_j \otimes \sigma w_l) \\ &= \#G \cdot \sum_{g \in G} h_3(v_i \otimes w_k, gv_j \otimes gw_l) \\ &= \#G \cdot \sum_{g \in G} h_1(v_i, gv_j) \cdot h_2(w_k, gw_l) \end{aligned} \quad (\text{C.4})$$

Now from (C.3) and the fact that $\{w_j\}_{j=1}^{2n}$ is a symplectic basis one has

$$\begin{aligned} h_2(w_k, gw_l) &= h_2(g^{-1}w_k, w_l) \\ &= \sum_{u=1}^{2n} T(g^{-1})_{u,k} h_2(w_u, w_l) \\ &= (-1)^l T(g^{-1})_{l-(-1)^l, k} \end{aligned} \tag{C.5}$$

Substituting (C.5) into (C.4) and then using (C.2) shows

$$h_3^G(\text{Tr}_G(v_i \otimes w_k), \text{Tr}_G(v_j \otimes w_l)) = (-1)^l \cdot \#G \cdot T(h(v_i, v_j))_{l-(-1)^l, k}. \tag{C.6}$$

The final equality in (C.6) gives (C.1).

Corollary C.4 Define $\psi^* h_3^G$ to be the alternating bilinear form on $V^{\perp q}$ which is the pull back via the isomorphism $\psi : V^q \rightarrow (M \otimes_F V)^G$ of Lemma C.1 of the bilinear form h_3^G on $(M \otimes_F V)^G$ defined in Definition C.2. For $x, y \in V^{\perp q}$ one has

$$\psi^* h_3^G(x, y) = h_2^{\perp q}(x, (-\#G) \cdot T(A)y)$$

where $(-\#G) \cdot T(A)$ is J -symmetric. The forms $\psi^* h_3^G$ and $h_2^{\perp q}$ are isometric (since they are alternating non-degenerate bilinear forms on the same vector space). Thus there is linear automorphism $P' : V^{\perp q} \rightarrow V^{\perp q}$ such that

$$P'^J \cdot P' = (-\#G) \cdot T(A).$$

Let χ be the character of V . Then

$$Pf((h(v_i, v_j))_{i,j})(\chi) = Pf^J(T(A)) = (-\#G)^{-q \dim(V)/2} Pf^J((-\#G) \cdot T(A)) = (-\#G)^{-q \dim(V)/2} \det(P')$$

for all such P' .

Proof: This is clear from Proposition C.3 and the definition of Pfaffians, together with the fact proved in [13, p. 40] that $Pf^J(a \cdot I) = a^{q \dim(V)/2}$ if $a \in \overline{\mathbf{Q}}$ and $I : V^{\perp q} \mapsto V^{\perp q}$ is the identity map. (Note that V has even dimension because it is symplectic.)

Remark C.5 The significance of Corollary C.4 is that it makes it possible to compute Pfaffians using two natural alternating bilinear forms on $(M \otimes_F V)^G$. This will be useful later when the forms on $(M \otimes_F V)^G$ can be written in a natural way as orthogonal sums.

D Appendix: Homotopies of perfect morphisms.

The object of this section is to prove Theorem 2.5, namely that $d(P^\bullet, \phi) = d(P^\bullet, \phi')$ if ϕ and ϕ' are homotopic perfect morphisms from $P_{\mathbf{Q}}^\bullet$ to $\widetilde{P_{\mathbf{Q}}^\bullet}$.

For simplicity of notation, let $M^\bullet = P_{\mathbf{Q}}^\bullet = P^\bullet \otimes_{\mathbf{Z}} \mathbf{Q}$. All morphisms $M^\bullet \rightarrow \widetilde{M^\bullet}$ which induce the same homomorphism on cohomology are homotopic. Hence it will suffice to prove Theorem 2.5

when ϕ is constructed as in the proof of Proposition 2.7(b) for $S_{\mathbf{Q}}^{\bullet} = M^{\bullet}$. Thus relative to a fixed G -direct sum decomposition

$$M^i = B^i(M^{\bullet}) \oplus H^i(M^{\bullet}) \oplus U^i$$

we can describe the pairings induced by ϕ in the following way. Suppose $i \neq 0$. Then the symmetric non-degenerate bilinear form $\langle \cdot, \cdot \rangle_{i,-i} : (M^i \oplus M^{-i}) \times (M^i \oplus M^{-i}) \rightarrow \mathbf{Q}$ induced by $\phi^i \oplus \phi^{-i} : M^i \oplus M^{-i} \rightarrow \widetilde{M}^{-i} \oplus \widetilde{M}^i$ is the orthogonal sum of three hyperbolic forms h_H^i , $h_{B,U}^i$ and $h_{B,U}^{-i}$, where h_H^i is a form on $H^i(M^{\bullet}) \oplus H^{-i}(M^{\bullet})$ and $h_{B,U}^i$ is a form on $B^i(M^{\bullet}) \oplus U^{-i}$. When $i = 0$, the form $\langle \cdot, \cdot \rangle_{0,0}$ on M^0 induced by $\phi^0 : M^0 \rightarrow \widetilde{M}^0$ is the orthogonal sum of h_H^0 and the hyperbolic form $h_{B,U}^0$.

Definition D.1 *Suppose V is a symplectic representation of G , with alternating non-degenerate G -invariant bilinear form $h_2 : V \times V \rightarrow \overline{\mathbf{Q}}$. If L is a $\mathbf{Q}G$ -module, let $L_V^G = (L \otimes_{\mathbf{Q}} V)^G$. If $f : L \times L \rightarrow \mathbf{Q}$ is a symmetric bilinear form, let $f_V^G : L_V^G \times L_V^G \rightarrow \overline{\mathbf{Q}}$ be the alternating bilinear form which is the restriction of $f \otimes h_2$ from $L \otimes_{\mathbf{Q}} V$ to L_V^G .*

In view of our description of the pairing $\langle \cdot, \cdot \rangle_{i,-i}$ as an orthogonal sum, we have the following result:

Lemma D.2 *Let V be as in Definition D.1, and \cdot . If $i \neq 0$ then $(\langle \cdot, \cdot \rangle_{i,-i})_V^G$ is the orthogonal sum*

$$((h_H)^i)_V^G \perp (h_{B,U}^i)_V^G \perp (h_{B,U}^{-i})_V^G.$$

If $i = 0$, then $(\langle \cdot, \cdot \rangle_{0,0})_V^G$ is the orthogonal sum

$$((h_H)^0)_V^G \perp (h_{B,U}^0)_V^G.$$

Proposition D.3 *Suppose $\phi^l : M^{\bullet} \rightarrow \widetilde{M}^{\bullet}$ is a perfect morphism homotopic to ϕ . Let $\delta^i : M^i \rightarrow M^{i+1}$ be the i -th boundary homomorphism of M^{\bullet} . There are isomorphisms $s_{i+1} : M^{i+1} \rightarrow M^i$ such that*

$$z_i = 1 + s_{i+1} \circ \delta^i + \delta^{i-1} \circ s_i \tag{D.1}$$

is an automorphism of M^i , and

$$\phi_{\mathbf{Q}}^i = \phi_{\mathbf{Q}}^i \circ z_i. \tag{D.2}$$

Proof: By definition, there are morphisms $k_{i+1} : M^{i+1} \rightarrow \widetilde{M}^i$ such that

$$\phi^i - \phi^i = k_{i+1} \circ \delta^i + \delta^{i-1} \circ k_i$$

where $\tilde{\delta}^{i-1} : \widetilde{M}^{i-1} \rightarrow \widetilde{M}^i$ is the $(i-1)$ -st boundary homomorphism. Because $\phi^i : M^i \rightarrow \widetilde{M}^i$ is an isomorphism, we can lift k_{i+1} to a homomorphism $s_{i+1} : M^{i+1} \rightarrow M^i$ such that $k_{i+1} = \phi^i \circ s_{i+1}$. This gives formula (D.2), and z_i must be an isomorphism because ϕ^i and ϕ^i are.

Proposition D.4 Define $M = M^i \oplus M^{-i}$ if $i \neq 0$, and let $M = M^0$ if $i = 0$. Let J be the adjoint involution of $\text{End}_{\overline{\mathbf{Q}}}(M_V^G)$ which is associated to the alternating bilinear form $(\langle, \rangle_{i,-i})_V^G$ described in Lemma D.2. Thus for $P \in \text{End}_{\overline{\mathbf{Q}}}(M_V^G)$ and $x, y \in M_V^G$ we have $\langle P^J x, y \rangle_V^G = \langle x, Py \rangle_V^G$. For $i \neq 0$ (resp. for $i = 0$) let $z = (z_i \oplus z_{-i})_V^G$ (resp. $z = (z_0)_V^G$) be the element of $\text{End}_{\overline{\mathbf{Q}}}(M_V^G)$ induced by $z_i \oplus z_{-i} : M^i \oplus M^{-i} \rightarrow M^i \oplus M^{-i}$ (resp. $z_0 : M^0 \rightarrow M^0$), where z_i is defined in (D.1). Then z is J -symmetric, i.e. $z^J = z$.

Proof: Suppose $i \neq 0$. Let the quadratic form

$$\langle, \rangle' = \langle, \rangle'_{i,-i} : (M^i \oplus M^{-i}) \times (M^i \oplus M^{-i}) \rightarrow \mathbf{Q}$$

be induced by $\phi^i \oplus \phi'^{-i} : M^i \oplus M^{-i} \rightarrow (M^{-i})^* \oplus (M^i)^*$. We have

$$\phi^i \oplus \phi'^{-i} = (\phi^i \oplus \phi^{-i}) \circ (z_i \oplus z_{-i})$$

This gives

$$\langle x, y \rangle_V^G = ((\phi^i \oplus \phi^{-i})_V^G \circ (z_i \oplus z_{-i})_V^G(x))(y) = \langle (z_i \oplus z_{-i})_V^G(x), y \rangle_V^G$$

Now both \langle, \rangle_V^G and \langle, \rangle'_V are alternating, so

$$\begin{aligned} \langle (z_i \oplus z_{-i})_V^G(x), y \rangle_V^G &= \langle x, y \rangle'_V \\ &= -\langle y, x \rangle'_V \\ &= -\langle (z_i \oplus z_{-i})_V^G(y), x \rangle_V^G \\ &= \langle x, (z_i \oplus z_{-i})_V^G(y) \rangle_V^G \end{aligned} \tag{D.3}$$

Since \langle, \rangle_V^G is non-degenerate, this shows $z = (z_i \oplus z_{-i})_V^G$ is J -symmetric. The argument when $i = 0$ is similar.

Corollary D.5 Suppose $\{v_l\}_{l=1}^q$ is a basis for the free $\mathbf{Q}G$ -module M . Let ϕ and ϕ' be as in Proposition D.3. Then ϕ defines a Hermitian form $h_M : M \times M \rightarrow \mathbf{Q}G$, and ϕ' defines a Hermitian form $h' : M \times M \rightarrow \mathbf{Q}G$. With the notations of Proposition D.4, one has

$$Pf^J(z) = \frac{Pf((h'(v_l, v_j))_{l,j})(\chi)}{Pf((h(v_l, v_j))_{l,j})(\chi)} \tag{D.4}$$

where χ is the character of V .

Proof: We use Corollary C.4 to compute $Pf((h'(v_l, v_j))_{l,j})(\chi)$ and $Pf((h(v_l, v_j))_{l,j})(\chi)$ by means of the alternating forms $(\langle, \rangle_{i,-i})_V^G$ and $(\langle, \rangle'_{i,-i})_V^G$ on M_V^G . The matrix of the form \langle, \rangle_V^G is equal to the matrix of the form \langle, \rangle'_V multiplied by the matrix of z_V^G . We now have (D.4) from the multiplicative properties of Pfaffians (c.f. [13, Prop. I.3.6]).

Proposition D.6 Suppose $L = \perp_{i=1}^n L_i$ is the orthogonal direct sum of hyperbolic spaces $L_i = L_{i,1} \oplus L_{i,2}$ over $\overline{\mathbf{Q}}$. Thus $L_{i,1}$ and $L_{i,2}$ are isotropic, and the alternating form on L_i identifies $L_{i,2}$ with $\widetilde{L_{i,1}} = \text{Hom}_{\mathbf{Q}}(L_{i,1}, \mathbf{Q})$. Suppose α is a symmetric automorphism of L which preserves the

spaces $W_k = \bigoplus_{i=1}^n L_{i,k}$ for $k = 1, 2$. We may then write the restriction α_k of α to W_k as a block matrix $(\alpha_k^{i,j})$, where $\alpha_k^{i,j} \in \text{Hom}_{\mathbf{Q}}(L_{j,k}, L_{i,k})$ and $\alpha_k|_{L_{j,k}} = \sum_{i=1}^n \alpha_k^{i,j}$. Then α_2 is the transpose $\widetilde{\alpha_1}$ of α_1 , so $\alpha_2^{j,i} = \widetilde{\alpha_1^{i,j}}$. Let $P : L \rightarrow L$ be the direct sum of $\alpha_1 : W_1 \rightarrow W_1$ and the identity map $W_2 \rightarrow W_2$. Then $P^J \cdot P = \alpha$, where J is the symplectic involution of L . Thus

$$Pf(\alpha) = \det(P) = \det(\alpha_1) = \det(P^J) = \det(\alpha_2).$$

Suppose finally that α_1 is upper triangular, in the sense that $\alpha_1^{i,j} = 0$ if $i > j$. Then α_2 is lower triangular. One has

$$\det(P) = \prod_{i=1}^n \det(\alpha_1^{i,i})$$

where

$$\det(\alpha_1^{i,i}) = \det(\widetilde{\alpha_1^{i,i}}) = \det(\alpha_2^{i,i}) = Pf(\alpha^i)$$

when $\alpha^i = \alpha_1^{i,i} \oplus \alpha_2^{i,i}$ is the symmetric automorphism of $L_i = L_{i,1} \oplus L_{i,2}$ which results from restricting α to L_i and then projecting onto L_i .

Proof: This is clear from the fact that if \langle, \rangle is the alternating form on L , then $\langle \alpha x_1, x_2 \rangle = \langle x_1, \alpha x_2 \rangle$ for $x_1 \in L_1$ and $x_2 \in L_2$.

Corollary D.7 *With the notation of Propositions D.4, let $B_i = B^i(M^\bullet)_V^G$, $U_i = (U^i)_V^G$ and let T_i be the summand $B^i \oplus U_{-i}$ of M_V^G . Let $z(T)$ be the endomorphism of a summand T of M_V^G which is induced by restricting z_V^G to T and then projecting from M_V^G to T . If $i \neq 0$, then*

$$Pf(z) = Pf(z(T_i)) \cdot Pf(z(T_{-i}))$$

where

$$Pf(z(T_j)) = \det(z(B_j)) = \det(z(U_{-j})).$$

If $i = 0$, then

$$Pf(z) = Pf(z(T_0)) = \det(z(B_0)) = \det(z(U_0)).$$

Proof Define $C^i = H^i(M^\bullet)_V^G \oplus H^i(M^\bullet)_V^G$ if $i \neq 0$ and $C^0 = H^0(M^\bullet)_V^G$. By Proposition D.6, we may write $L = M_V^G$ as the orthogonal sum of $L_1 = T_i$, $L_2 = C^i$ and $L_3 = T_{-i}$ (resp. $L_1 = T_0$ and $L_2 = C_0$) if $i \neq 0$. Each of these L_j is given as a hyperbolic space except for C_0 , and in that case we simply choose any hyperbolic basis for C_0 . The formula for z in (D.1) of Proposition D.3 shows the hypotheses of Proposition D.6 are satisfied since $s_{i+1} \circ \delta^i$ is 0 on $B^i(M^\bullet) \oplus H^i(M^\bullet)$ and $\delta^{i-1} \circ s_i$ has image in $B^i(M^\bullet)$. In particular, z induces the identity map on $B^i(M^\bullet)_V^G \oplus H^i(M^\bullet)_V^G$ modulo $B^i(M^\bullet)_V^G$, so $z(H^i(M^\bullet)_V^G)$ is the identity. Hence $Pf(z(H^i(M^\bullet)_V^G)) = 1$, and the statements in Corollary D.7 now follow from Proposition D.6.

Proof of Theorem 2.5.

Suppose ϕ' is a perfect morphism from M^\bullet to \widetilde{M}^\bullet which is homotopic to our fixed ϕ . Following the recipe for computing $d(P^\bullet, \phi)$ and $d(P^\bullet, \phi')$ which is given in Definitions 2.4 and 4.4, we see that to prove $d(P^\bullet, \phi) = d(P^\bullet, \phi')$ it will suffice to show that the Pfaffians associated to a fixed choice of local and global bases for the terms of P^\bullet do not change when one replaces ϕ by ϕ' . Corollaries D.5 and D.7 show that if V is a given symplectic representation of G , the factor by which the Pfaffians change when one replaces ϕ by ϕ' can be written as a product of the form

$$\det(z(B_0)) \cdot \prod_{i < 0} (\det(z(B_i)) \cdot \det(z(U_i)))^{(-1)^i} \quad (\text{D.5})$$

where the terms in this expression are defined in Corollary D.7. The boundary homomorphism $\delta^i : M^i \rightarrow M^{i+1}$ induces an isomorphism between U^i and $B^{i+1}(M^\bullet)$ which identifies the automorphism $z(U_i)$ of U_i with the automorphism $z(B_{i+1})$ of B_{i+1} . Thus the product in (D.5) telescopes to equal 1. This completes the proof.

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