MATH 703: HOMEWORK #1

DUE WEDNESDAY, JAN. 23, 2013

1. Hensel's Lemma

This problem is about a generalization of Hensel's Lemma to polynomials in two variables. Let K be a p-adic field with integers O_K and absolute value $|\ | : K \to \mathbb{R}$ normalized so that $|\pi_K| = q^{-1}$ when π_K is a uniformizer in O_K and $q = \#O_K/(\pi_K O_K)$. Suppose $f_1(x,y), f_2(x,y) \in O_K[x,y]$ are polynomials in two variables over O_K. We then have a polynomial map $K \times K \to K \times K$ defined by

$$(x,y) \to F(x,y) = (f_1(x,y), f_2(x,y)).$$

Suppose $(x_0,y_0) \in O_K \times O_K$ has the property that

$$F(x_0,y_0) \in (\pi_K O_K) \times (\pi_K O_K).$$

1. State and prove a generalization of the naive version of Hensel's Lemma which will provide a sufficient condition for there to exist $(x_1,y_1) \in O_K \times O_K$ such that
$$F(x_1,y_1) = (0,0) \quad \text{and} \quad (x_1,y_1) \equiv (x_0,y_0) \mod (\pi_K O_K) \times (\pi_K O_K).$$
2. Apply your criterion in problem #1 to the case in which $(x_0,y_0) = (1,-1)$, $f_1(x,y) = x^3 + xy + \pi_K$ and $f_2(x,y) = x^2 - y^2 - \pi_K$.
3. State and prove a generalization of the sophisticated form of Hensel's Lemma based on Newton's iteration.

2. Extensions of absolute values

Let p be a prime and let $\overline{\mathbb{Q}}_p$ be an algebraic closure of $\mathbb{Q}_p$. We will sketch in class a proof that there is a unique non-archimedean absolute value $|\ |_p : \overline{\mathbb{Q}}_p \to \mathbb{R}$ which extends the usual p-adic absolute value $|\ |_p : \mathbb{Q}_p \to \mathbb{R}$.

4. Show that if $\alpha \in \overline{\mathbb{Q}}_p$ and $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ then $|\sigma(\alpha)|_p = |\alpha|_p$.
5. Suppose that $f(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_0 \in \mathbb{Q}_p$ for some $n \geq 2$ and $b_i \in \mathbb{Z}_p$. If $b_0 \in \mathbb{Z}_p$ and $b_j \notin \mathbb{Z}_p$ for some $0 < j < n$, is it possible that $f(x)$ is irreducible in $\mathbb{Q}_p[x]$?