MATH 620: HOMEWORK #3

1. Disjoint extensions with coprime disciminants

This problem generalizes Proposition 17 of Chapter 3 of Lang's "Algebraic Number Theory" book.

Suppose L and N are two finite separable extensions of a field F inside an algebraic closure $\overline{F}$ of F. We will say that L and N are disjoint over F if whenever $\{l_i\}_i$ is a basis for L over F and $\{w_j\}_j$ is a basis for N over F, the set $\{l_i w_j\}_{i,j}$ is a basis for the compositum LN over F.

Let A be a Noetherian subring of F such that $F = \mathrm{Frac}(A)$ and A is integrally closed in F. If T is a field such that $F \subset T \subset LN$, let A_T be the integral closure of A in T, and let $D(A_T/A) \subset A$ be the discriminant ideal of A_T over A. We will use without further comment the fact that if S is a multiplicatively closed subset of A, then $S^{-1}A_T$ is the integral closure of $S^{-1}A$ in T and $D(S^{-1}A_T/S^{-1}A) = S^{-1}D(A_T/A)$.

We will say that A_L and A_N have coprime discriminants over A if for each prime ideal P of A, either

$$(A - P)^{-1}D(A_L/A) = (A - P)^{-1}A = A_P$$

or

$$(A - P)^{-1}D(A_N/A) = (A - P)^{-1}A = A_P.$$

The object of this exercise is to show:

Theorem 1.1. *If L and N are disjoint finite separable extensions of F, and A_L and A_N have coprime discriminants over A, then the integral closure A_{LN} of A in LN is the subring $A_L \cdot A_N$ generated by A_L and A_N.*

1. Show the conclusion of the Theorem will follow if we show

$$(A - P)^{-1}(A_L \cdot A_N) = (A - P)^{-1}A_{LN}$$

for all primes P of A. Explain why we can then reduce to the case in which A is a local ring and either $D(A_L/A) = A$ or $D(A_N/A) = A$.

2. Suppose A is a local ring and that $D(A_N/A) = A$. Recall that $D(A_N/A)$ is the A-ideal generated by all disciminants $D(\{w_j\}_j)$ of bases $\{w_j\}_j$ for N over F such that $\{w_j\}_j \subset A_N$. Show that there is one such basis $\{w_j\}_j$ which spans the same A-module as it's dual basis $\{w_\ell^*\}_\ell$, and that A_N is the direct sum $\oplus_j Aw_j$.

3. Show that if $\{w_j\}_j$ is as in problem # 2, then a basis for LN as an L-vector space is given by $\{w_j\}_j$. Use $\{w_\ell^*\}_\ell$ and the trace from LN to N to show that if $\beta = \sum_j \beta_j w_j$ lies in A_{LN} for some $\beta_j \in L$, then $\beta_j \in A_L$. Deduce Theorem 1.1 from this.

4. Show that if L/F and N/F are finite Galois extensions, then L and N are disjoint over F if and only if $L \cap N = F$. Is this still true if we drop the assumption that L/F and N/F are Galois?

2. The Carlitz module

Let p be a prime, $L = \mathbb{F}_p(t)$ and $A = \mathbb{F}_p[t]$. In class we discuss the Carlitz module defined by the ring homomorphism $\psi : A \to L\{\tau\}$ sending t to $t + \tau$, where $L\{\tau\}$ is the twisted polynomial ring for which $\tau\beta = \beta^p\tau$ for $\beta \in L$. Then $L\{\tau\}$ acts on an algebraic closure $\overline{L}$ of L by letting $\beta \in L$ act by multiplication by β, and by letting τ send $\alpha \in \overline{L}$ to $\tau(\alpha) = \alpha^p$. If $\pi(t) \in A$ is not 0, define the $\pi(t)$-torsion subgroup of $\overline{L}$ by

$$\mu_{\pi(t)} = \{\alpha \in \overline{L} : \psi(\pi(t))(\alpha) = 0\}$$

5. Suppose $\pi(t) \in A = \mathbb{F}_p[t]$ is monic of degree $d \geq 1$ in t. Show that $\mu_{\pi(t)}$ is the set of all roots of a separable polynomial of degree p^d, and that $\mu_{\pi(t)}$ is an additive group.

6. With the notation of problem # 5, show that there is an action of the ring $A/\pi(t)A$ on $\mu_{\pi(t)}$ induced by letting the class of $h(t) \in A$ send $\alpha \in \mu_{\pi(t)}$ to $\psi(h(t))(\alpha)$. Show that this makes $\mu_{\pi(t)}$ into a free rank one module for $A/\pi(t)A$. (To prove freeness, it may be useful to factor $\pi(t)$ into a product of powers of distinct irreducibles $r(t)$ and to consider the size of $\mu_{r(t)} \subset \mu_{\pi(t)}$.)

Comment: This fact corresponds to the statement that multiplicative group of all roots of $x^n - 1$ in $\mathbb{C}$ is a free rank 1 module for the ring $\mathbb{Z}/n$.

7. Suppose $\pi(t)$ is a monic irreducible polynomial of degree d. Let $\alpha \in \mu_{\pi(t)}$ be a generator for $\mu_{\pi(t)}$ as a free rank one module for the field $A/A\pi(t)$. Try showing that the integral closure of $B = \mathbb{F}_p[t]$ in the field $L(\mu_{\pi(t)})$ obtained by adjoining to L all elements of $\mu_{\pi(t)}$ is the ring $B[\alpha]$ generated by B and α. In doing this, it may be useful to construct an analog of the proof that $\mathbb{Z}[\zeta_p]$ is the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\zeta_p)$.