MATH 620: HOMEWORK #1

1. Review of some algebra

There problems review some ideas from first year graduate algebra.

1. Suppose R is a commutative ring. Show that the polynomial ring $R[X]$ is a principal ideal domain if and only if R is a field.

2. If R is a Noetherian ring, is every subring of R Noetherian? Prove this, or give a counterexample.

3. Suppose R is a Noetherian integral domain with fraction field K. A fractional ideal of R is defined to be a non-zero finitely generated R-submodule of K. The generic rank of an R-module M is defined to be the dimension over K of the K vector space $K \otimes_R M$. The torsion of M is the kernel of the homomorphism $M \to K \otimes_R M$ defined by $\alpha \to 1 \otimes \alpha$.

3a. Show that the fractional ideals of R have the form $x^{-1}I$ for some $x \in R - \{0\}$ and some non-zero ideal $I \subset R$.

3b. Show that a finitely generated R-module M has torsion $\{0\}$ and rank 1 if and only if M is isomorphic to a fractional ideal. Is this true if we drop the condition that M be finitely generated as an R-module?

2. Integral elements

4. Suppose R is a possibly non-commutative ring. Suppose A is a subring of the center of R, so that every element of A commutes with every element of R. One can then say that $x \in R$ is integral over A if there is an integer $n \geq 1$ and $a_i \in A$ such that

$$x^n + a_{n-1}x^{n-1} + \ldots + a_0 = 0.$$

4a. Show that if R is finitely generated as an A-module, every $x \in R$ is integral over A. (Hint: Review the proof when R is commutative.)

4b. Show that if $x \in R$ is integral over A, then uxu^{-1} is also integral for all units $u \in R^*$.

4c. Must it be the case that the set R' of $x \in R$ which are integral over A forms a subring of R? Prove this or give a counterexample.

5. Suppose A is an integral domain which is integrally closed in its fraction field K in the sense that A is its own integral closure in K. Suppose $q \in A$ is not a square in K, so that $L = K(\sqrt{q}) = K + K\sqrt{q}$ is a quadratic extension of K. Describe the conditions on $r, s \in K$ which are necessary and sufficient for $\alpha = r + s\sqrt{q} \in L$ to be in the integral closure A' of A in L. Check that this gives the description discussed in class of the ring $A' = O_L$ of integers of $L = \mathbb{Q}(\sqrt{q})$ when $A = \mathbb{Z}$.

3. Transcendence

This problem has to do with a counterpart for Laurent series of Liouville's Theorem. Recall that the classical version of Liouville's theorem is:

Theorem 3.1. *Suppose $\alpha \in \mathbb{R}$ is an algebraic number of degree $\leq n$, in the sense that*

$$\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_0 = 0 \tag{3.1}$$

for some integer $n \geq 1$ and some rationals $a_i \in \mathbb{Q}$. Then for all constants $c, \epsilon > 0$, there are only finitely many rationals $\frac{p}{q}$ with $p, q \in \mathbb{Z}$ and $q \neq 0$ such that

$$|\alpha - \frac{p}{q}| < \frac{c}{q^{n+\epsilon}} \tag{3.2}$$

The proof of this Theorem proceeds by first clearing the denominators of the a_i in (3.1) to produce a polynomial

$$F(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_0 \tag{3.3}$$

with integer coefficients b_i such that $b_n \neq 0$ and $F(\alpha) = 0$. If $\frac{p}{q}$ is a rational number which is not one of the (finitely many) roots of $F(x)$, we get the estimate

$$|F(\frac{p}{q})| = |b_n(\frac{p}{q})^n + \cdots b_0| = \frac{|b_n p^n + b_{n-1}p^{n-1}q + \cdots + b_0 q^n|}{q^n} \geq \frac{1}{q^n} \tag{3.4}$$

since $b_n p^n + b_{n-1}p^{n-1}q + \cdots + b_0 q^n$ is a non-zero integer. On the other hand, if $|\frac{p}{q} - \alpha| \leq c$, we have from the Mean Value Theorem that there is a number λ between α and $\frac{p}{q}$ such that

$$|F(\frac{p}{q})| = |F(\frac{p}{q}) - F(\alpha)| = |F'(\lambda)| \cdot |\frac{p}{q} - \alpha| \leq M|\frac{p}{q} - \alpha| \tag{3.5}$$

where

$$M = \sup\{|F'(\lambda)| \ : \ \alpha - c \leq \lambda \leq \alpha + c\}.$$

Combining (3.4) and (3.5) gives

$$|\frac{p}{q} - \alpha| \geq \frac{1}{M \cdot q^n} \quad \text{if} \quad |\frac{p}{q} - \alpha| \leq c$$

and this leads to the conclusion of Liouville's Theorem, since M depends only on $F(x)$, α and c.

We now develop a counterpart of Liouville's Theorem for the field $k((x))$ of formal Laurent series in one variable over a field k. Recall that $k((x))$ is the field of all formal series

$$f(x) = \sum_{n=N}^{\infty} a_n x^n \tag{3.6}$$

in which N is a (possibly negative) integer, the a_n are in k, and addition and multiplication are done formally. The set all such $f(x)$ for which $N \geq 0$ forms the power series ring $k[[x]]$. (You should think through why $k((x))$ is a field which contains the field $k(x)$ of all rational functions in x over k.)

6. Suppose r is a real number and $0 < r < 1$. Define a function $| \ | : k((x)) \to \mathbb{R}$ by setting $|0| = 0$ and by letting

$$|f(x)| = r^N$$

when $f(x)$ is as in (3.6) and $a_N \neq 0$. Show this is a non-archimedean norm, in the sense that for all $f(x), g(x) \in k((x))$,

6a. $|f(x) \cdot g(x)| = |f(x)| \cdot |g(x)|$

6b. $|f(x) + g(x)| \leq \max(|f(x)|, |g(x)|)$

7. Observe that $k(x)$ is the same as the field $k(x^{-1})$ of rational functions in x^{-1}. In the arguments in later parts of this problem, the ring $k[x^{-1}]$ plays the same role relative to the field $k((x))$ as $\mathbb{Z}$ does relative to $\mathbb{R}$ in the classical version of Liouville's Theorem. To begin with, show that if $0 \neq f(x) \in k[x^{-1}]$ then $|f(x)| \geq 1$.

8. The counterpart of Liouville's Theorem we will prove is the following. Suppose that $u = u(x) \in k((x))$ is algebraic over $k(x)$ of degree $\leq N$, in the sense that it satisfies an equation

$$u^N + a_{N-1}u^{N-1} + \cdots + a_0 = 0 \tag{3.7}$$

for some integer $N \geq 1$ and some rational functions $a_i = a_i(x) \in k(x) \subset k((x))$.

Theorem 3.2. *For all constants $c, \epsilon > 0$ there is a constant $\delta = \delta(c, \epsilon, u) > 0$ depending on c, ϵ and u for which the following is true. If $p, q \in k[x^{-1}]$, $q \neq 0$ and*

$$|u - \frac{p}{q}| \le \frac{c}{|q|^{N+\epsilon}} \tag{3.8}$$

then $|q| \le \delta$.

Assuming this result for the moment, use it to prove that $u(x) = \sum_{n=0}^{\infty} x^{n!}$ is an element of $k((x))$ which is transcendental over $k(x) = k(x^{-1})$. Can one say strengthen the Theorem to say that there are only finitely many $\frac{p}{q}$ for which (3.8) holds?

9. As a first step toward proving Theorem 3.2, write each $a_i = a_i(x)$ as a ratio of s_i/r_i of elements $s_i, r_i \in k[x^{-1}]$. Show that u is a root of a polynomial

$$F(X) = b_N X^N + \cdots + b_0$$

in which the b_i are in $k[x^{-1}]$ and $b_N \neq 0$. Now adjust the classical proof of Liouville's theorem using the properties of $| \ |$ shown in problem # 6. The key step is to prove that there is a real number M which depends only on $|u|$, c, N and on the b_i such that

$$|F(\frac{p}{q})| = |F(\frac{p}{q}) - F(u)| \le M \cdot |\frac{p}{q} - u| \tag{3.9}$$

if $|\frac{p}{q} - u| \le c$ and $p, q \in k[x^{-1}]$. To prove this bound, expand

$$F(\frac{p}{q}) - F(u) = \sum_{i=0}^{N} b_i((\frac{p}{q})^i - u^i)$$

using the binomial theorem and apply problem # 6.